Potential Energy of Two Collinear Cracks

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The energetics of two unequal-length collinear cracks is considered in Mode I, II, III loadings. The material force, i.e., the energy change accompanying the translation of each crack, the expanding moment accompanying their isotropic expansion, and the total potential energy of the system are evaluated. The analysis is performed for remote uniform load normal to the cracks plane using the concept of the J and M path-independent integrals. The closed form exact solution to this interaction problem is obtained as a function of cracks dimensions and their spacing. While both the material force and the moment tend to infinity as the distance between the inner crack tips approaches zero, the total strain energy of cracks converges to the limit. The material force decays rapidly to zero as the two cracks become a few lengths apart. Comparisons of the obtained results with those for two equal cracks and with other available solutions are considered and discussed.

1. Introduction

Many models for the average effective properties of elastic solids are based on the approximation of non-interacting inhomogeneities which utilize various elastic solutions for an isolated crack. We consider here the strain energy of two unequal cracks that could lead to a deeper understanding of pair interaction effects of microcracks. The energy required to form one crack has been found in analytical form for many various geometrical arrangements and load conditions. In the case of two equal collinear Griffith cracks, in an isotropic material, opened by uniform pressure, the analytical solution in a closed form was obtained by Willmore [1] in 1949. Lowengrub and Srivastava [2] treated an infinitely long layer containing two equal collinear cracks located parallel to its surfaces and opened by an arbitrary internal pressure, by reducing the problem to the Fredholm integral equation. The analytical expression for the energy at relatively large distance between cracks was obtained. Adams [3] solved the problem of a layer containing an arbitrary number of unequal-size cracks. The problem was reduced to singular integral equations and the crack energy was determined by quadratures.

When the pressure is constant, the energy in the aforementioned cases of two or more cracks has been determined by integration of the shape of the deformed cracks over the crack faces. Along with the direct integration approach, the crack energy can be treated by different methods, including the application of the J- and

M- integrals. For instance, Budiansky and O'Connell [4] used the relations between the energy and these integrals when treating the energy of an elliptic crack. In this paper, we calculate the energy of two cracks located in infinite elastic medium that is subjected to remote uniform load, invoking the relationship [5, 6] between the energy, J- and M- integrals. In case of cracks, the J- and M- integrals can be evaluated as long as the solution for the stress intensity factors (SIF) is available. The problem of the energy then can be reduced to calculation of the SIFs.

The interaction of two unequal-size collinear cracks for the case of uniaxial tension normal to the line of cracks was supposedly first studied by Panasyuk and Lozoviy [7]. Applying the Muskhelishvili method, they determined the stress intensity factors (SIF) for all four crack tips and expressed them in terms of the complete elliptic integrals of the first kind, third kind, and an additional not tabulated integral. The SIFs obtained from this solution are also presented in graphical form by Rooke and Cartwright [8]. Applying the concept of continuous arrays of infinitesimal dislocations, Yokobori et al. [9] reduced the two-crack problem to the integral equation and solved it. The closed-form solution was presented only for the higher values of SIFs, i.e., for the inner crack tips, and they were expressed in terms of the complete elliptic integrals of the first kind, K, and the second kind, E. Their results are available in several publications, see, for instance, Ref. 10 and the references therein. In this paper we complete the Yokobori et. al. solution providing the SIFs for outer tips of collinear cracks, compute the J- and M- integrals, and obtain the potential energy of two cracks in the closed analytical form.

2. Stress Intensity Factors for Two Unequal-length Cracks

Consider an infinite elastic solid with two collinear asymmetrical cracks subjected to a perpendicularly applied normal stress σ at infinity, which causes the body to deform in a state of plane strain as shown in Fig. 1.

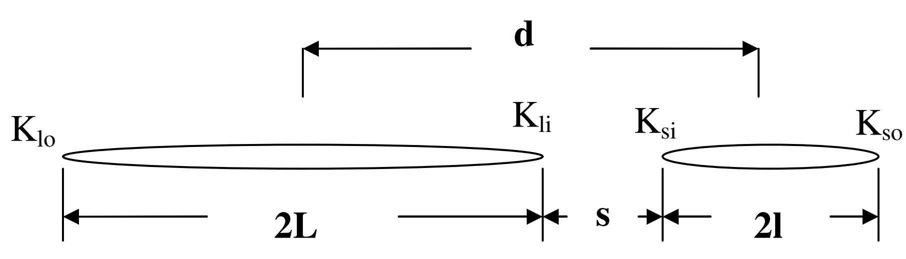

Fig. 1: Two collinear asymmetrical cracks.

Let us call the left crack with length $2L$ - the first crack, and the right crack with the length $2l$ - the second crack. It is supposed that s is the distance of separation between the inside tips and d is the distance between midpoints of the cracks. The crack geometry can be described by a pair of non-dimensional parameters

$$\alpha = \frac{L}{l} \quad , \qquad\qquad \beta = \frac{s}{2l} \tag{1}$$

Similar to the procedure used by Yokobori et al. [9] for the inner crack tips, i, we derive the SIFs for outer tips, o, from their solution of the integral equation (8) by substituting the corresponding coordinates and taking necessary limits. We normalize all the obtained expressions with respect to the value of the SIF of a single Griffith crack of length $2l$.

$$K_s = \sigma\sqrt{\pi\ell} \tag{2}$$

Marking the crack tips by the following subscripts: lo - large crack, outer tip; li – large crack, inner tip; so - small crack, outer tip; si - small crack, inner tip, we arrive at the expressions in the unified non-dimensional form for all the SIFs that can be written as:

$$F_{lo} = \frac{K_{lo}}{K_s} = \frac{\sqrt{\alpha+\beta}\,[(\alpha+\beta+1)-(\beta+1)\,e(k)]}{\sqrt{\alpha(\alpha+\beta+1)}} \tag{3}$$

$$F_{li} = \frac{K_{li}}{K_s} = \frac{\sqrt{1+\beta}\;\;[\beta-(\alpha+\beta)\,e(k)]}{\sqrt{\alpha\beta}} \tag{4}$$

$$F_{si} = \frac{K_{si}}{K_s} = \frac{\sqrt{\alpha+\beta}\;\;[\beta-(1+\beta)\,e(k)]}{\sqrt{\beta}} \tag{5}$$

$$F_{so} = \frac{K_{so}}{K_s} = \frac{\sqrt{1+\beta}\;\;[(\alpha+\beta+1)-(\alpha+\beta)\,e(k)]}{\sqrt{\alpha+\beta+1}} \quad , \tag{6}$$

where:

$$k = \sqrt{\frac{\alpha}{(\alpha+\beta)(1+\beta)}} \;; \tag{7}$$

$$e(k) = \frac{E(k)}{K(k)}$$

is the ratio of the complete elliptic integral of the second kind, E, to the complete elliptic integrals of the first kind, K.

Some observations can be made concerning these equations. In equations (4,5) we see that the SIFs for the inner tips, reached at s=0, become infinite while for the outer tips, as shown in equations (3,6), they are still finite as the two cracks approach each other. In this case, two cracks form one large crack of length 2 (l+L) and the values of SIFs for the outer tips are equal to those of the formed single crack which takes the following form in non-dimensional notations

$$F_{lo} = F_{so} = \sqrt{1+\alpha} \tag{8}$$

In the limit, when the distance of crack separation, s, becomes large, the obtained values of SIFs agree with the corresponding values for a single crack of length 2l or 2L, respectively.

When both cracks have an equal size, equations (3-6) simplify and due to the symmetry they reduce to two equations. These equations can be transformed exactly to those derived by Willmore [1, 10] and, thus, we recover his results for two equal cracks. Willmore expressed the SIFs through the crack tip coordinates, and therefore for the transformation of results the conversion property of the elliptical integrals given in Ref. [11] should be applied.

With the SIFs given by Eqs. (3-6), the driving forces, G_j, for each crack tip are determined by the Irwin formula

$$G_j = \frac{1-\nu^2}{E} K_j^2, \tag{9}$$

where E is the Young modulus of elasticity and ν is the Poisson ratio. The subscript j runs over all crack tips: lo, li, so, si.

3. Relation between the crack energy, J- and M-integrals of Fracture Mechanics

The path-independent J-and M- integrals are related to potential energy changes, or energy release rates, as a cavity translates or expands isotropically in solids. The classical J-integral and the M-integral are defined as follows:

$$J_k = \int_c \left(Wn_k - n_j \sigma_{ji} u_{i,k}\right) d\ell \tag{10}$$

$$M = \int_c x_k \left(Wn_k - n_j \sigma_{ji} u_{i,k}\right) d\ell \tag{11}$$

where, in two-dimensional space, C is a closed curve surrounding the cavity, W is the strain energy density, u_k and σ_{ji} are the components of the displacement and stress field, *n* is the unit outward normal to C, and *dl* is the infinitesimal arc length along the curve C. J and M can be interpreted as the negative of the potential energy release rate, as a cavity undergoes a unit translation in certain direction or isotropic expansion relative to the origin, and they are called the material force and the material scalar (or expanding) moment, respectively. If the curve C encloses a system of inhomogeneities, the material force of (10), J_{tot} , describes the change of the total potential energy, *U*, of the entire system of inhomogeneities due to a unit translation. Assume that a system of inhomogeneities is inserted into the homogeneous stress field, then the energy of this system will be the same wherever the position of the entire system of inhomogeneities is and the total material force on the entire system vanishes, $J_{\mathrm{tot}} = 0$.

According to this, the material resultant force for our system of two cracks inserted into the uniform stress field, i.e.,. an infinite plane elastic domain subjected to constant remote stress, should vanish. This fact has been confirmed by direct vector summation of all four material crack driving forces upon substitution of (3-6) into (9). Then it leads to

$$J_{tot} = J_1 + J_2 = 0$$

The choice of the contour C in (11) that engulfs the entire system of inhomogeneities results in the total scalar moment, M_{tot} . In the case of two inhomogeneities inserted into the homogeneous stress field, the following relation [5,6] between the potential energy, U, and J- and M-integrals holds

$$M_{tot} = 2U = M_1 + M_2 + \boldsymbol{r} \cdot \mathbf{J_1} \tag{12}$$

Here, the M_1 and M_2 are integrals calculated for each inhomogeneity, the $\mathbf{J}_1$ is the material force on the 1st inhomogeneity due to mutual interaction between them; $\boldsymbol{r}$ stands for the position vector of the 1st inhomogeneity with respect to the 2nd inhomogeneity.

In general, the M-integral depends on the choice of a reference point. If the M-integral is evaluated in one case with origin at $\mathbf{r}_p$ and in another with origin at $\mathbf{r}_q$, then [12]

$$M_p = M_q + (\mathbf{r}_q - \mathbf{r}_p) \cdot \mathbf{J} \tag{13}$$

However, when $\boldsymbol{J} = 0$, it is obvious from (13) that the M –integral is independent of a choice of reference point and this statement is also true for equation (12), when $J_{\mathrm{tot}} = 0$.
In our case of the system of two cracks, we can express the J- and M- integrals by the crack driving forces as follows

$$J_1 = G_{\ell i} - G_{\ell o} \tag{14}$$

$$M_1 = (G_{\ell i} + G_{\ell o})L \tag{15}$$

$$M_2 = (G_{si} + G_{so})\ell \tag{16}$$

As the two cracks approach each other, the SIFs for the inner cracks tips become infinite, both the material force (14) and the scalar moments (15,16) tend to infinity as the distance between the inner crack tips approaches zero. As the two cracks become a few lengths apart, it can be shown that the mutual interaction force decays rapidly to zero.

Introducing the force vector driving a crack tip, $\mathbf{G}_j$, the position vector of the crack tip, $\mathbf{r}_j$, and substituting (14-16) into equation (12), we arrive at the energy expression in a simple form

$$U = \mathbf{G}_j \cdot \mathbf{r}_j \tag{17}$$

where j is running over all four crack tips: *li, lo, si, so.*

4. Evaluation of the potential energy

The crack energy can be computed by substituting (3-6) into (9) and, subsequently, substituting the result into equation (17). After some tedious but straightforward algebraic manipulations, we arrive at the following expression for the energy of two cracks:

$$U = \frac{\pi\sigma^2 d^2\left(1-\nu^2\right)}{E}\left\{1-\left[1-(\lambda_1-\lambda_2)^2\right]\cdot e(k)\right\} \tag{18}$$

$$\text{where} \quad \lambda_1 = \frac{L}{d}; \qquad \lambda_2 = \frac{\ell}{d}; \tag{19}$$

k was already defined through the non-dimensional parameters α and β by equation (7). For convenience, it is rewritten here with these two new non-dimensional parameters given in (19).

$$k = 2\sqrt{\frac{\lambda_1\lambda_2}{1-(\lambda_1-\lambda_2)^2}} \tag{20}$$

Applying the series expansion of e(k) over k [11] and performing an asymptotic study of the series as d→∞, it can be verified that in the limit as the distance of separation between the midpoints becomes large, equation (18) yields the expected result for the energy of two single isolated cracks.

In the case of two equal cracks, $\lambda_1 = \lambda_2$, or α=1, U assumes a simple form as a function of $\lambda = \lambda_1 + \lambda_2 = \frac{2\ell}{d}$:

$$U = \frac{4\pi\sigma^2 \ell^2\left(1-\nu^2\right)}{E}\left[1-e(\lambda)\right] \tag{21}$$

We can recover the results of Willmore [1] for the energy of two equal cracks by applying the conversion property of the elliptic integrals [11] to equation (21). The transformation of our results is similar to that mentioned previously for the SIF transformation.

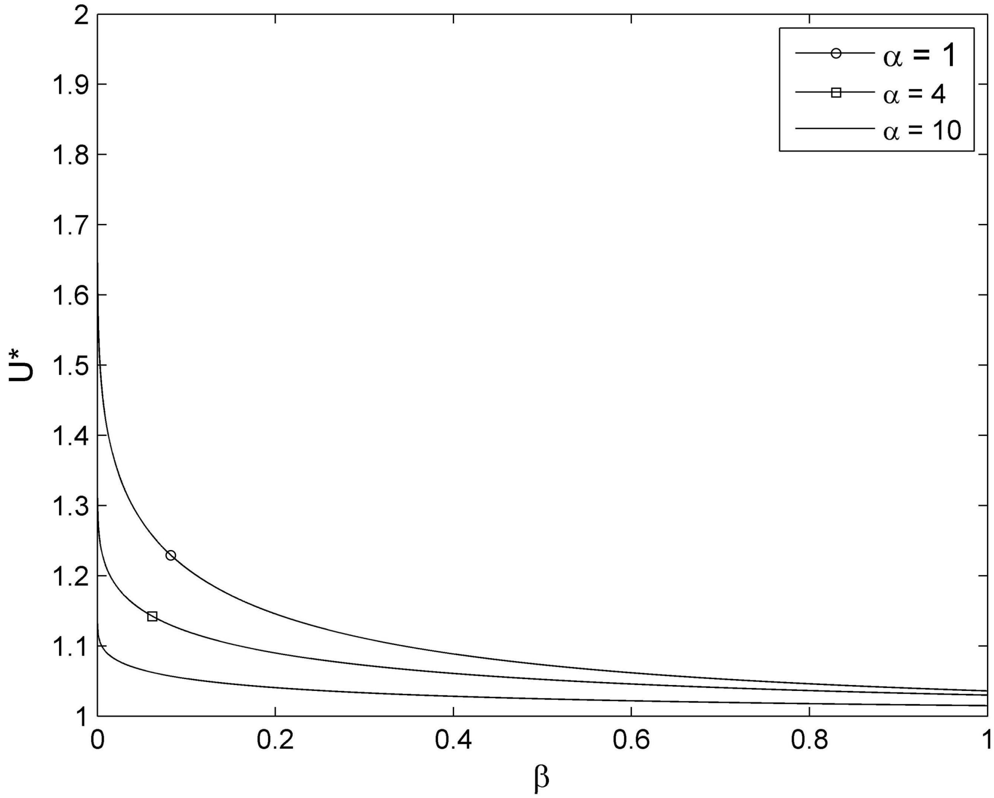

Fig. 2. Non-dimensional crack energy, U*, as a function of the crack separation distance, β, at α = 1, α = 4 and α = 10.

To demonstrate the contribution of the energy of mutual interaction between the cracks to the total energy, some numerical results are presented in Figure 2 for different crack lengths at non-dimensional ratios of crack lengths: α =1, 4, and 10. We normalize U with respect to the total strain energy of both cracks when they are isolated. The graph for the non-dimensional strain energy,

$$U^* = \frac{UE}{\pi\sigma^2\left(\ell^2 + L^2\right)\left(1-\nu^2\right)} \tag{22}$$

is plotted as a function of the non-dimensional distance of separation of two cracks, β.

It is seen that U* grows as s tends to zero. In the limiting case of small separation, U has a bound. As s → 0, we can readily show that

$$U^* = \frac{(1+\alpha)^2}{1+\alpha^2} \tag{23}$$

This solution is valid for tension loading condition or crack opening Mode I. It is also applicable to in-plane shear, Mode II, and anti-plane shear, Mode III when the loading conditions are shear stress, τ, remote from the cracks. All the above results are valid with σ replaced by τ. For the Mode III, the term $(1-\nu^2)$ should be also replaced by $(1+\nu)$.

5. Discussion

The problem of two collinear cracks for remote uniform load normal to the cracks plane is analyzed. Applying the concept of the J and M path-independent integrals, the exact solution to this interaction problem is obtained as a function of the cracks dimensions and their spacing. As the distance between the inner crack tips approaches zero, the total strain energy of cracks converges to the limit. Comparisons of the obtained results with those for two equal cracks and other cases are presented.

The obtained results can be implemented for evaluation of effective properties of elastic solids. When the applied load is uniform and constant, the total potential energy is related to the change in the corresponding compliance of damaged solids because the energy is proportional to the total crack opening area of both cracks. So, as long as the solution for the SIF is known, the crack energy and the change in elastic moduli can be estimated. Most of the models for the ensemble average properties are based on the approximation of non-interacting inhomogeneities and an analytical treatment of the strain energy for two unequal cracks could provide more insight into the nature of microcrack interaction, especially in the case when pair interaction of near cracks is considered.

It should be noted that equation (12) can be extended to the case of several cavities and formula (17) is actually applicable to an arbitrary number of collinear cracks. The approach of this work is neither limited to the number of cracks, nor to the infinite solid domain. It can be extended to some other configurations for the system of two and more cracks. For instance, when provided with the values of SIFs for the problems considered in Ref. 3, the total crack energy can be calculated based on equation (17). Our results show good agreement with those presented therein. However, if the state of applied stress is not homogeneous, either due to non-uniform loadings or due to the presence of a boundary or interface, an analysis and results would depend on the validity of equation (12) or (17). The generalizations and limitations of the present approach will be discussed elsewhere.

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