Crack theory with possible contact between the crack faces

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It is well known that classical linear crack problems in solid mechanics are characterized by linear boundary conditions imposed at the crack faces. Such linear models allow the opposite crack faces to penetrate each other which leads to inconsistency with practical situations. Since the beginning of 1990, the crack theory with non-penetration conditions is under active study. This theory is characterized by inequality type boundary conditions at the crack faces. The book [1] contains results for models with non-penetrations for different constitutive laws, i.e. 2D and 3D models as well as plate and shell models with inequality type boundary conditions are analyzed. After publication of this book, new approaches and trends in study of non-linear crack models with the non-penetration have been developed. For example, a problem of differentiation of the energy functional with respect to crack perturbations is solved in a general setting, smooth and fictitious domain methods are proposed, invariant integrals are constructed in cases of different geometrical situations, etc. We refer the reader to publications [2], [3], [4], [5], [6], [7], [8]. Moreover, it turned out that many practical problems should be described by crack models with non-penetration for overlapping domains. For example, the overlapping approach is applicable for description of a subduction phenomenon of tectonic plates, a slipping phenomenon of ice plates, construction of complicated precise level devices, etc. Considering suitable structures, we actually may consider Riemann surfaces with two or more sheets having cracks with inequality type boundary conditions at the crack faces [9], [10]. Similar structures can be also found in animate nature. For instance, fish scales can be seen as multi-layer structure which needs an overlapping domain approach for their description. In engineering practice, structures like "patches" at the crack tips also can be used, i.e, we obtain overlapping domain description. In the talk, some last results obtained in this nonlinear crack theory are discussed, including overlapping domain approach.

Problem formulation. Let $\Omega \subset R^2$ be a bounded domain with smooth boundary Γ, and $\gamma \subset \Omega$ be a smooth curve without self-intersections, $\Omega_\gamma = \Omega \setminus \overline{\gamma}$. It is assumed that γ can be extended in such a way that this extension crosses Γ at two points, and Ω is divided into two subdomains D_1 and D_2 with Lipschitz boundaries ∂D_1, ∂D_2, $meas(\Gamma \cap \partial D_i) > 0$, $i = 1, 2$. Denote by $\nu = (\nu_1, \nu_2)$ a unit normal vector to γ. We assume that γ does not contain its tip points, i.e. $\gamma = \overline{\gamma} \setminus \partial\gamma$. Equilibrium problem for a linear elastic body occupying Ω_γ is as follows. In the domain Ω_γ we have to find a displacement field $u = (u_1, u_2)$ and stress tensor components $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, such that

$$-\mathrm{div}\sigma = f \quad \text{in} \quad \Omega_\gamma, \tag{1}$$

$$\sigma = A\varepsilon(u) \quad \text{in} \quad \Omega_\gamma, \tag{2}$$

$$u = 0 \quad \text{on} \quad \Gamma, \tag{3}$$

$$[u]\nu \geq 0, \quad [\sigma_\nu] = 0, \quad \sigma_\nu \cdot [u]\nu = 0 \quad \text{on} \quad \gamma, \tag{4}$$

$$\sigma_\nu \leq 0, \quad \sigma_\tau = 0 \quad \text{on} \quad \gamma^\pm. \tag{5}$$

Here $[v] = v^+ - v^-$ is a jump of v on γ, and signs $\pm$ correspond to positive and negative crack faces with respect to ν, $f = (f_1, f_2) \in L^2(\Omega_\gamma)$ is a given function,

$$\sigma_\nu = \sigma_{ij}\nu_j\nu_i, \quad \sigma_\tau = \sigma\nu - \sigma_\nu \cdot \nu, \quad \sigma_\tau = (\sigma_\tau^1, \sigma_\tau^2),\ \sigma\nu = (\sigma_{1j}\nu_j, \sigma_{2j}\nu_j),$$

the strain tensor components are denoted by $\varepsilon_{ij}(u)$,

$$\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \varepsilon(u) = \{\varepsilon_{ij}(u)\}, \quad i, j = 1, 2.$$

Elasticity tensor $A = \{a_{ijkl}\}$, $i, j, k, l = 1, 2$, is given, and it satisfies the usual properties of symmetry and positive definiteness

$$a_{ijkl}\xi_{kl}\xi_{ij} \geq c_0|\xi|^2, \quad \forall\ \xi_{ij},\ \xi_{ij} = \xi_{ji}, \quad c_0 = const > 0,$$

$a_{ijkl} = a_{klij} = a_{jikl}$, $a_{ijkl} \in L^\infty(\Omega)$.

The first condition in (4) is called the non-penetration condition. It provides a mutual non-penetration between the crack faces $\gamma^\pm$. The second condition of (5) provides zero friction on γ. For simplicity we assume a clamping condition (3) at the external boundary Γ.

Note that a priori we do not know points on γ where strict inequalities in (4), (5) are fulfilled. Due to this, (1)-(5) is a free boundary value problem. If we have $\sigma_\nu = 0$ then, together with $\sigma_\tau = 0$, the classical boundary condition $\sigma\nu = 0$ follows which is used in the linear crack theory. On the other hand, due to (4), the condition $\sigma_\nu < 0$ implies $[u]\nu = 0$, i.e. we have a contact between the crack faces at a given point. The strict inequality $[u]\nu > 0$ at a given point means that we have no contact between the crack faces.

The problem (1)-(5) is well posed. Therefore, there is a unique weak solution to the associated variational inequality. We introduce also the so-called smooth domain formulation which is equivalent to (1)-(5).

First of all we note that problem (1)-(5) corresponds to minimization of the energy functional. To check this, introduce the Sobolev space

$$H^1_\Gamma(\Omega_\gamma) = \{v = (v_1, v_2) \mid v_i \in H^1(\Omega_\gamma),\ v_i = 0 \text{ on } \Gamma,\ i = 1, 2\}$$

and closed convex set of admissible displacements

$$K = \{v \in H^1_\Gamma(\Omega_\gamma) \mid [v]\nu \geq 0 \quad \text{a.e. on} \quad \gamma\}. \tag{6}$$

In this case the problem

$$\min_{v\in K}\left\{\frac{1}{2}\int\limits_{\Omega_\gamma}\sigma_{ij}(v)\varepsilon_{ij}(v) - \int\limits_{\Omega_\gamma} f_i v_i\right\}$$

has (a unique) solution u satisfying the variational inequality

$$u \in K, \tag{7}$$

$$\int\limits_{\Omega_\gamma}\sigma_{ij}(u)\varepsilon_{ij}(v-u) \geq \int\limits_{\Omega_\gamma} f_i(v_i - u_i) \quad \forall v \in K, \tag{8}$$

where $\sigma_{ij}(u) = \sigma_{ij}$ are defined from (2).

Problem formulations (1)-(5) and (7)-(8) are equivalent. Any smooth solution of (1)-(5) satisfies (7)-(8) and conversely, from (7)-(8) it follows (1)-(5).

Below we provide one more equivalent formulation for the problem (1)-(5), the so-called smooth domain formulation. To this end, we first discuss in what sense boundary conditions (4)-(5) are fulfilled. Denote by Σ a closed curve without self-intersections of the class $C^{1,1}$, which is an extension of γ such that $\Sigma \subset \Omega$, and the domain Ω is divided into two subdomains Ω_1 and Ω_2. In this case Σ is the boundary of the domain Ω_1, and the boundary of Ω_2 is $\Sigma \cup \Gamma$.

Introduce the space $H^{\frac{1}{2}}(\Sigma)$ with the norm

$$\|v\|^2_{H^{\frac{1}{2}}(\Sigma)} = \|v\|^2_{L^2(\Sigma)} + \int\limits_\Sigma\int\limits_\Sigma \frac{|v(x)-v(y)|^2}{|x-y|^2}dxdy \tag{9}$$

and denote by $H^{-\frac{1}{2}}(\Sigma)$ a space dual of $H^{\frac{1}{2}}(\Sigma)$. Also, consider the space

$$H^{1/2}_{00}(\gamma) = \left\{v \in H^{\frac{1}{2}}(\gamma) \mid \frac{v}{\sqrt{\rho}} \in L^2(\gamma)\right\}$$

with the norm

$$\|v\|^2_{1/2,00} = \|v\|^2_{1/2} + \int\limits_\gamma \rho^{-1}v^2,$$

where $\rho(x) = dist(x; \partial\gamma)$, and $\|v\|_{1/2}$ is the norm in the space $H^{1/2}(\gamma)$. With the above notations, it is possible to describe in what sense boundary conditions (4)-(5) are fulfilled. Namely, the condition $\sigma_\nu \leq 0$ in (5) means that

$$\langle \sigma_\nu, \phi \rangle_{1/2,00} \leq 0 \quad \forall\, \phi \in H^{1/2}_{00}(\gamma), \quad \phi \geq 0 \text{ a.e. on } \gamma,$$

where $\langle \cdot, \cdot \rangle_{1/2,00}$ is a duality pairing between $H^{-1/2}_{00}(\gamma)$ and $H^{1/2}_{00}(\gamma)$. The condition $\sigma_\tau = 0$ in (5) means that

$$\langle \sigma^i_\tau, \phi \rangle_{1/2,00} = 0 \quad \forall\, \phi = (\phi_1, \phi_2) \in H^{1/2}_{00}(\gamma), \quad i = 1, 2, \;\; \phi_i \nu_i = 0.$$

The last condition of (4) holds in the following sense

$$\langle \sigma_\nu, [u]\nu \rangle_{1/2,00} = 0.$$

Smooth domain formulation. Notice that the solution of the problem (1)-(5) satisfies (7)-(8), thus the condition

$$[\sigma\nu] = 0 \quad \text{on} \quad \gamma$$

holds, and therefore it can be proved that in the distributional sense

$$-\text{div}\sigma = f \quad \text{in} \quad \Omega.$$

Hence, the equilibrium equations (1) hold in the smooth domain Ω.

Introduce the space for stresses defined in Ω,

$$\mathcal{H}(\text{div}) = \{\sigma = \{\sigma_{ij}\} \mid \sigma, \ \text{div}\sigma \in L^2(\Omega)\}$$

and the set of admissible stresses

$$\mathcal{H}(\text{div}; \gamma) = \{\sigma \in \mathcal{H}(\text{div}) \mid \sigma_\tau = 0, \quad \sigma_\nu \leq 0 \text{ on } \gamma\}.$$

We see that for $\sigma \in \mathcal{H}(\text{div})$, the boundary condition $\sigma_\tau = 0$, $\sigma_\nu \leq 0$ on γ are correctly defined in the sense of $H^{-1/2}_{00}(\gamma)$. Thus, we can provide the smooth domain formulation for the problem (1)-(5). It is necessary to find a displacement field $u = (u_1, u_2)$ and stress tensor components $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, such that

$$u \in L^2(\Omega), \quad \sigma \in \mathcal{H}(\text{div}; \gamma), \tag{10}$$

$$-\text{div}\sigma = f \quad \text{in} \quad \Omega, \tag{11}$$

$$\int\limits_\Omega C\sigma(\overline{\sigma} - \sigma) + \int\limits_\Omega u(\text{div}\overline{\sigma} - \text{div}\sigma) \geq 0 \quad \forall \overline{\sigma} \in \mathcal{H}(\text{div}; \gamma). \tag{12}$$

Here the tensor $C = \{c_{ijkl}\}$ is obtained by inverting the Hooke's law (2), i.e. $C\sigma = \varepsilon(u)$. It is possible to prove a solution existence to the problem (10)-(12). Moreover, any smooth solution of (1)-(5) satisfies (10)-(12) and conversely, from (10)-(12) it follows (1)-(5), i.e. (1)-(5) and (10)-(12) are equivalent. Advantage of the formulation (10)-(12) is that it is given in the smooth domain. This formulation reminds contact problems with thin obstacle when restrictions are imposed on sets of small dimensions ([11]).

Fictitious domain method. We can provide a connection between the problem (1)-(5) and the Signorini contact problem. It is turned out that the Signorini problem is a limit problem for a family of problems like (1)-(5). First, we remind a formulation of the Signorini problem. Let $\Omega_1 \subset \mathbb{R}^2$ be a bounded domain with smooth boundary Γ_1, $\Gamma_1 = \gamma \cup \Gamma_0$, $\gamma \cap \Gamma_0 = \emptyset$, $meas\Gamma_0 > 0$. For simplicity, we assume that γ is a smooth curve (without its tip points). Denote by $\nu = (\nu_1, \nu_2)$ a unit normal inward vector to γ. We have to find a displacement field $u = (u_1, u_2)$ and stress tensor components $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, such that

$$-\text{div}\sigma = f \quad \text{in} \quad \Omega_1, \tag{13}$$

$$\sigma = A\varepsilon(u) \quad \text{in} \quad \Omega_1, \tag{14}$$

$$u = 0 \quad \text{on} \quad \Gamma_0, \tag{15}$$

$$u\nu \geq 0,\ \sigma_\nu \leq 0,\ \sigma_\tau = 0,\ u\nu \cdot \sigma_\nu = 0 \quad \text{on} \quad \gamma. \tag{16}$$

Here $f = (f_1, f_2) \in L^2_{loc}(R^2)$ is a given function, $A = \{a_{ijkl}\}$, $i, j, k, l = 1, 2$, is a given elasticity tensor, $a_{ijkl} \in L^\infty_{loc}(\mathbb{R}^2)$, with the usual properties of symmetry and positive definiteness.

The problem (13)-(16) has a variational formulation. Namely, denote

$$H^1_{\Gamma_0}(\Omega_1) = \{v = (v_1, v_2) \in H^1(\Omega_1) \mid v_i = 0 \text{ on } \Gamma_0, \quad i = 1, 2\}$$

and introduce the set of admissible displacements

$$K_c = \{v = (v_1, v_2) \in H^1_{\Gamma_0}(\Omega_1) \mid v\nu \geq 0 \text{ a.e. on } \gamma\}.$$

In this case the problem (13)-(16) is equivalent to minimization of the functional

$$\frac{1}{2}\int\limits_{\Omega_1} \sigma_{ij}(v)\varepsilon_{ij}(v) - \int\limits_{\Omega_1} f_i v_i$$

over the set K_c and can be written in the form of variational inequality

$$u \in K_c, \tag{17}$$

$$\int\limits_{\Omega_1} \sigma_{ij}(u)\varepsilon_{ij}(v-u) \geq \int\limits_{\Omega_1} f_i(v_i - u_i) \quad \forall v \in K_c. \tag{18}$$

Here $\sigma_{ij}(u) = \sigma_{ij}$ are defined from the Hooke's law (14). Variational inequality (17)-(18) is equivalent to (13)-(16). It is possible to prove that the problem (13)-(16) is a limit problem for a family of problems like (1)-(5).

In what follows we provide some comments on this score. First of all we extend the domain Ω_1 by adding a domain Ω_2 with smooth boundary Γ_2. An extended domain is denoted by Ω_γ, and it has a crack (cut) γ. Boundary of Ω_γ is $\Gamma \cup \gamma^\pm$. Denote $\Sigma_0 = \Gamma_1 \cap \Gamma_2$, $\Sigma = \Sigma_0 \setminus \Gamma$, thus Σ does not contain its tip points.

We introduce a family of elasticity tensors with a positive parameter λ,

$$a_{ijkl}^\lambda = \begin{cases} a_{ijkl} & \text{in } \Omega_1 \\ \lambda^{-1} a_{ijkl} & \text{in } \Omega_2. \end{cases}$$

Denote $A^\lambda = \{a_{ijkl}^\lambda\}$, and in the extended domain Ω_γ, consider a family of the crack problems. Find a displacement field $u^\lambda = (u_1^\lambda, u_2^\lambda)$, and stress tensor components $\sigma^\lambda = \{\sigma_{ij}^\lambda\}$, $i, j = 1, 2$, such that

$$-\mathrm{div}\sigma^\lambda = f \quad \text{in} \quad \Omega_\gamma, \tag{19}$$

$$\sigma^\lambda = A^\lambda \varepsilon(u^\lambda) \quad \text{in} \quad \Omega_\gamma, \tag{20}$$

$$u^\lambda = 0 \quad \text{on} \quad \Gamma, \tag{21}$$

$$[u^\lambda]\nu \geq 0, \ [\sigma_\nu^\lambda] = 0, \ \sigma_\nu^\lambda \cdot [u]\nu = 0 \quad \text{on} \quad \gamma, \tag{22}$$

$$\sigma_\nu^\lambda \leq 0, \ \sigma_\tau^\lambda = 0 \quad \text{on} \quad \gamma^\pm. \tag{23}$$

As before, $[v] = v^+ - v^-$ is a jump of v through γ, where $\pm$ fit positive and negative crack faces $\gamma^\pm$. We see that for any fixed $\lambda > 0$ the problem (19)-(23) describes an equilibrium state of linear elastic body with the crack γ where non-penetration conditions are prescribed. Hence, the problem (19)-(23) is exactly the problem like (1)-(5), and we are interested in passage to the limit as $\lambda \to 0$. In particular, the problem (19)-(23) admits a variational formulation. It can be proved that the following convergence takes place as $\lambda \to 0$

$$u^\lambda \to u^0 \quad \text{strongly in} \quad H_\Gamma^1(\Omega_\gamma), \tag{24}$$

$$\frac{u^\lambda}{\sqrt{\lambda}} \to 0 \quad \text{strongly in} \quad H^1(\Omega_2), \tag{25}$$

where $u^0 = u$ on Ω_1, i.e. a restriction of the limit function from (24) to Ω_1 coincides with the unique solution of the Signorini problem (13)-(16). From (24)-(25) it is seen that the limit function u^0 is zero in Ω_2. On the other hand, generally speaking, there is no limit of σ^λ in Ω_2 as $\lambda \to 0$. Thus, the domain Ω_2 can be understood as non-deformable body. This means that the

Signorini problem is, in fact, a crack problem with non-penetration condition between crack faces, where the crack γ is located between the elastic body Ω_1 and non-deformable body Ω_2. It is worth noting that we can write the problem (19)-(23) in the equivalent form in the smooth domain $\Omega_\gamma \cup \overline{\gamma}$ by using the smooth domain formulation.

Overlapping domain problem. Again, let $\Omega \subset R^2$ be a bounded domain with smooth boundary Γ. Assume that the set $\gamma = (0,1) \times \{0\}$ belongs to Ω. Consider a neighborhood $\omega \subset \Omega$ of the point $(1,0)$ with smooth boundary $\partial\omega$ assuming that $(1,0)$ does not belong to Γ. Let $\nu = (\nu_1, \nu_2)$ be a unit normal vector to γ, and $n = (n_1, n_2)$ be outward normal unit vector to $\partial\omega$. In our considerations, domain Ω corresponds to an elastic body with the crack γ, and ω fits for another elastic body. The body ω can be viewed as elastic "patch" imposed in the neighborhood ω of the crack tip $(1,0)$. In further analysis, rigidity properties of ω would depend on a parameter δ. For any fixed rigidity parameter we can find the derivative of the energy functional with respect to the crack length which is related to the Griffith criterion of crack propagations. Allowing the rigidity of the body ω going to infinity we derive the limit problem for the body Ω_γ with the crack γ and a rigid "patch". This particular problem is also analyzed from the standpoint of differentiability of energy functional. The principal interest concerns relations between the energy derivatives for the limit problem and the problem for the body with the elastic "patch". The main result obtained states that derivatives converge as $\delta \to 0$.

Consider tensors of elasticity $A = \{a_{ijkl}\},\ B = \{b_{ijkl}\},\ i,j,k,l = 1,2,$ with usual properties of symmetry and positive definiteness. For simplicity we assume that a_{ijkl}, b_{ijkl} are constants. Let δ be a positive parameter. Problem formulation of the equilibrium problem for the two elastic bodies with the glue set $\partial\omega$ is as follows. We have to find functions $u^\delta(x) = (u_1^\delta(x), u_2^\delta(x)),\ v^\delta(z) = (v_1^\delta(z), v_2^\delta(z)),\ x \in \Omega_\gamma,\ z \in \omega$, such that

$$-\mathrm{div}\sigma^\delta = f \ \text{ in } \ \Omega_\gamma \setminus \partial\omega, \tag{26}$$

$$\sigma^\delta = A\varepsilon(u^\delta) \ \text{ in } \ \Omega_\gamma, \tag{27}$$

$$-\mathrm{div}p^\delta = 0 \ \text{ in } \ \omega, \tag{28}$$

$$p^\delta = \frac{1}{\delta}B\varepsilon(v^\delta) \ \text{ in } \ \omega, \tag{29}$$

$$u^\delta = 0 \ \text{ on } \ \Gamma, \tag{30}$$

$$[u^\delta]\nu \geq 0,\ [\sigma_\nu^\delta] = 0,\ \sigma_\nu^\delta \leq 0,\ \sigma_\tau^\delta = 0,\ \sigma_\nu^\delta \cdot [u^\delta]\nu = 0 \ \text{ on } \ \gamma, \tag{31}$$

$$u^\delta = v^\delta,\ [\sigma^\delta n] = p^\delta n \ \text{ on } \ \partial\omega. \tag{32}$$

Here $\sigma^\delta = \{\sigma^\delta_{ij}\}$, $p^\delta = \{p^\delta_{ij}\}$ are stress tensors for the first and the second bodies, respectively, $i, j = 1, 2$; $\sigma^\delta_\tau = \sigma^\delta \nu - \sigma^\delta_\nu \cdot \nu$, $\sigma^\delta \nu = (\sigma^\delta_{1j}\nu_j, \sigma^\delta_{2j}\nu_j)$, $\sigma^\delta_\nu = \sigma^\delta_{ij}\nu_j\nu_i$.

The Hooke law (29) depends on δ. We first consider the problem (26)-(32) for any fixed δ, and next analyze a passage to the limit as $\delta \to 0$. On the glue set $\partial\omega$ it is necessary to impose boundary conditions (32) for $\delta > 0$.

Note that equilibrium equations (26) hold in $\Omega_\gamma \setminus \partial\omega$, and simultaneously, the Hooke law (27) holds in Ω_γ.

We can justify a passage to the limit as $\delta \to 0$ in (26)-(32).

For the problem (26)-(32) it is possible to find the derivative G^δ of the energy functional with respect to the crack length. Also, it is possible to find the derivative G^0 of the energy functional with respect to the crack length for the limit problem corresponding $\delta = 0$. Moreover we can prove that $G^\delta \to G^0$, see [10].

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