A VARIATIONAL APPROACH OF FATIGUE DEBONDING

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ABSTRACT

We study the debonding of a thin layer initially glued to a rigid substrate and submitted at one end to a constant tension and a cyclic deflexion. The theoretical framework of the modelling is the variational approach of fracture proposed first by Francfort and Marigo [1] and extended here in order to model the propagation of cracks by fatigue. We adopt a Dugdale surface energy, we introduce an irreversibility condition and we require, as in [1], that the body minimizes, at each loading step, its total energy. With these ingredients we obtain fatigue laws like those usually postulated by engineers. The number of cycles until the total debonding depends in particular on the ratio ε between the internal length appearing in Dugdale energy and the length of the layer. When ε goes to 0, we show that the limit fatigue law is a Paris law; the "rate of debonding growth per cycle" is a function of the energy release rate $\dot{\ell} = f(G)$.

1 INTRODUCTION

Francfort and Marigo [1] propose a variational approach of brittle fracture in which the cracks appear and grow in an elastic brittle body in such a manner that the total energy of the body is minimal at each step of the loading history. In that work, the authors conserve Griffith's hypothesis by assuming that the surface energy is proportional to the surface area of the crack independently of the value of the displacement jump discontinuity. But, with this choice, it is impossible to render account for fatigue phenomenon. We propose in the present work to extend this approach in order to model the propagation of cracks in bodies submitted to cyclic loadings. The idea is to conserve the principle of least energy, but to replace the Griffith-type surface energy by an energy depending on the displacement jump through the crack and to introduce an irreversibility condition. The ingredients and the method are developed in the case of the fatigue debonding of a thin layer.

2 SETTING OF THE PROBLEM

2.1 The ingredients

The problem is 2D. We consider an inextensible and perfectly flexible thin layer of length L, which is perfectly glued to a rigid substrate at the initial time. One end of the layer is submitted to a constant tension $N\,\mathbf{e}_1$ and a cyclic deflexion $V(t)\,\mathbf{e}_2$; the other end is fixed, cf. Figure 1. We denote by $\mathbf{U}(s,t) = u(s,t)\,\mathbf{e}_1 + v(s,t)\,\mathbf{e}_2$ the displacement of the layer point of abscissa s at time t. The components u and v verify the kinematical boundary conditions: $u(0,t) = v(0,t) = 0$ and $v(L,t) = V(t)$. The prescribed deflexion $V(t)$ is periodic in time, varying from 0 to a maximal value V_m, see Figure 2. Moreover, since the film is inextensible and perfectly flexible, the potential energy can be written in terms of the deflexion field v and reads as:

$$\mathsf{P}(v) = \frac{N}{2}\int_0^L v'(s)^2 ds \quad . \tag{1}$$

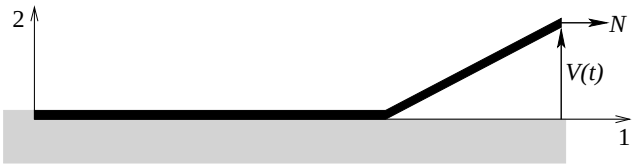

Figure 1. Geometry

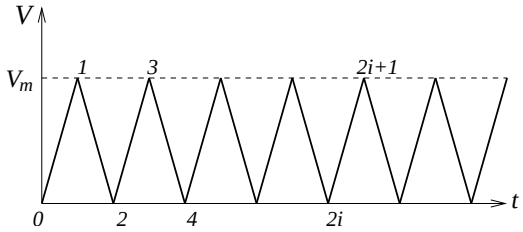

Figure 2. Cyclic loading

To take into account the debonding irreversibility, we introduce a memory field δ, called "the cumulated opening", which accounts for the accumulation with time of the debonding between the film and the substrate. At time t and point s, $\delta(s,t)$ reads as

$$\delta(s,t) = \int_0^t \langle \dot{v}(s,\tau) \rangle d\tau \qquad \text{with} \qquad \langle f \rangle = \max\{0, f\}. \tag{2}$$

Once the loading path is discretized, the index i refers to the time step and relation (2) becomes

$$\delta_i(s) = \delta_{i-1}(s) + \langle v_i(s) - v_{i-1}(s) \rangle . \tag{3}$$

The density surface energy Φ is taken as the following function of δ (Dugdale surface energy):

$$\Phi(\delta) = \begin{cases} G_c \dfrac{\delta}{\delta_c} & \text{if} \quad 0 \le \delta \le \delta_c \\ G_c & \text{if} \quad \delta \ge \delta_c \end{cases} . \tag{4}$$

In eqn (4), δ_c represents an internal characteristic length and G_c the toughness of the interface. The surface energy of the interface whose cumulated opening field is δ reads then as

$$\mathsf{S}(\delta) = \int_0^L \Phi(\delta(s)) ds. \tag{5}$$

2.2 The incremental problem

Following the idea first introduced by Francfort and Marigo [1] in the context of Griffith's theory, then developed by Del Piero [2], Charlotte *et al*. [3], Truskinovsky and Marigo [4] for Barenblatt surface energy, the evolution of the debonding will be given by minimizing the total energy of the structure, sum of its potential energy (1) and of its surface energy (5). However, because of the irreversibility condition, we must first discretize the loading path and then perform the minimization at each discrete step of the loading. That leads to a sequence of minimization problems, the solution at step i depending on the solutions at the previous steps. Specifically, denoting by V_i the set of admissible deflexions at step i,

$$\mathsf{V}_i = \left\{ v \in W^{1,2}(0,L) : \quad v(0) = 0, \quad v \geq 0, \quad v(L) = V(t_i) \right\}, \tag{6}$$

and by $\mathsf{E}_i(v)$ the total energy of the layer at this step for a deflexion v,

$$\mathsf{E}_i(v) = \mathsf{P}(v) + \mathsf{S}\left(\delta_{i-1} + \langle v - v_{i-1} \rangle\right), \tag{7}$$

the incremental problem reads as

For $i \in \mathbb{N}$ *Find* $v_{i+1} \in \mathsf{V}_{i+1}$ *and* δ_{i+1}

such that

$$\mathsf{E}_{i+1}(v_{i+1}) = \underset{v \in \mathsf{V}_{i+1}}{Min}\, \mathsf{E}_{i+1}(v), \qquad \delta_{i+1} = \delta_i + \langle v_{i+1} - v_i \rangle$$

with the initial condition $v_0 = \delta_0 = 0$.

Remark 1 : We can prove that this incremental problem admits a unique solution and that the solution is independent of the discretization. Consequently, each loading or unloading part of a cycle can be treated in one step. That allows us to consider from now that the index i refers to the ends of a half-cycle as in Figure 2. Moreover the debonding does not evolve during the unloading loading steps and the layer returns to its initial displacement configuration at the end of an unloading step: $v_{2i} = 0$ *and* $\delta_{2i} = \delta_{2i-1}$.

3 EVOLUTION OF THE DEBONDING

The minimization is performed at each step by using both classical and direct methods of Calculus of Variations. The main properties of the minimizer are presented in this section.

3.1 Evolution of the crack tips

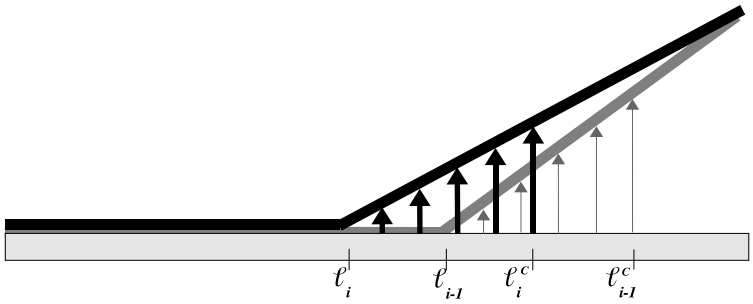

Figure 3. Positions of the crack tip and of the process zone tip at the end of the loading phase of the cycles *i-1* and *i*.

Because of Barenblatt surface energy a process zone appears in front of the crack. In this zone, the layer is not completely debonded but submitted to cohesive forces given by the derivative of the surface energy with respect to the cumulated opening. In the case of Dugdale surface energy, this cohesive force is constant and equal to G_c / δ_c inside the process zone. At the end of the loading phase of the i[th] cycle, the process zone is located in the interval $(\ell_i L,\ \ell_i^c L)$ whereas the first part of the layer $(0,\ \ell_i L)$ is still perfectly bonded and the end of the layer $(\ell_i^c L,\ L)$ is entirely

debonded. When $V_m \geq \delta_c$, the position of the process zone tip (fictitious crack tip) ℓ_i and of the "true" crack tip ℓ_i^c are given by the following sequence of two equations:

$$\begin{cases} \left(\ell_i^c - \ell_i\right)^2 + 2\left(\ell_i^c - \ell_i\right)\left(1 - \ell_i^c\right) = 4\varepsilon \overline{V}_m \\ \sum_{j=1}^{i} \left\langle \ell_i^c - \ell_j \right\rangle^2 = 4\varepsilon^2 \end{cases} . \tag{8}$$

In eqn (8), $\overline{V}_m$ denotes the dimensionless amplitude of the cycle whereas ε is a dimensionless parameter linked to the ratios between the Dugdale internal length δ_c and the overall length L of the layer, on one hand, and, between the prescribed tension N and the toughness G_c, on the other hand:

$$\overline{V}_m = \sqrt{\frac{N}{2G_c}}\frac{V_m}{L}, \qquad \varepsilon = \sqrt{\frac{N}{2G_c}}\frac{\delta_c}{L}. \tag{9}$$

So the debonding evolution depends on the two parameters ε and $\overline{V}_m$. The system (8) is solved by induction. In particular, when $\overline{V}_m \geq \varepsilon$, at the end of the first loading step, the process zone tip and the true debonding tip are respectively located at

$$\ell_1 = 1 - \varepsilon - \overline{V}_m, \qquad \ell_1^c = 1 + \varepsilon - \overline{V}_m. \tag{10}$$

For the next cycles a numerical treatment is necessary. In Figures 4 and 5 are plotted the evolutions of the true debonding tip versus the cycle number for various values of the parameters ε and $\overline{V}_m$.

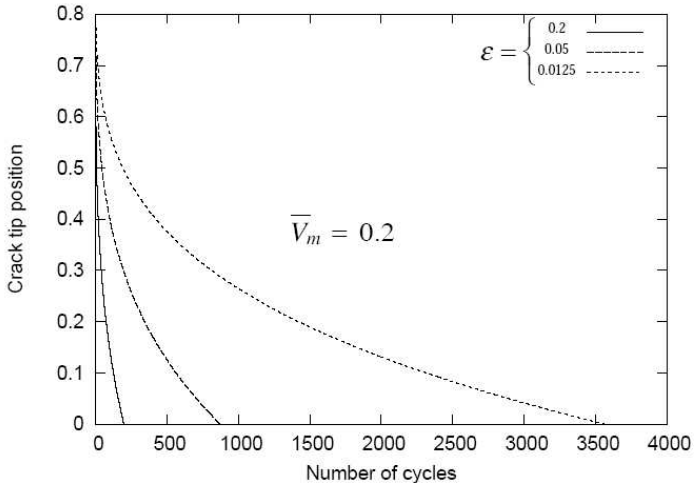

Figure 4. Influence of the parameter ε on the debonding evolution

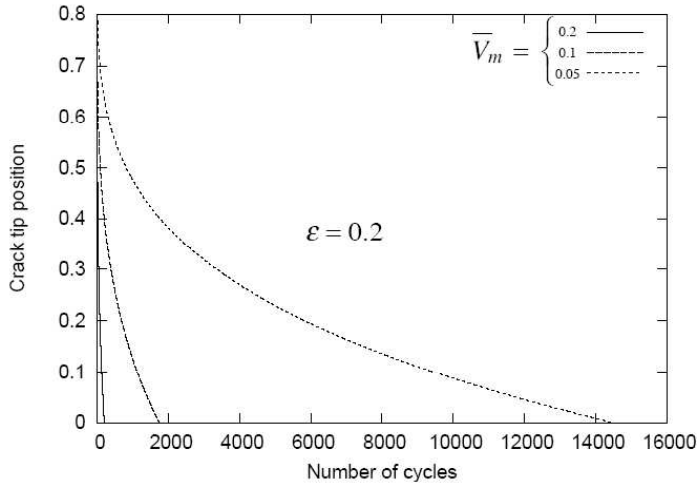

Figure 5. Influence of the amplitude of loading on the debonding evolution

Remark 2 : It can be proved that, for given values of ε *and* $\overline{V}_m$, $\ell_i^c \to 0$ *when* $i \to \infty$*; the layer tends to be entirely debonded when the number of cycles tends to infinity.*

3.2 The Limit Fatigue Law when $\varepsilon \to 0$

Solving the two equations (8) requires considerable computational time if we consider low values of $\overline{V}_m$ or ε. In particular the increment of the propagation of the process zone per cycle, $\ell_i^c - \ell_{i-1}^c$, is of order ε. In the context of a layer length L large in comparison with the characteristic length of the surface energy δ_c, *i.e.* when $\varepsilon \to 0$, the number of cycle until a complete debonding of the layer tends to infinity like $1/\varepsilon$. So if we rescale the number of cycles by introducing the continuous "time" variable T and putting

$$i = i_\varepsilon(T) = \frac{T}{\varepsilon}, \tag{11}$$

then we can see on Figure 6 that the curve $\ell^c_{i_\varepsilon(T)}$ representing the evolution of the true debonding tip converges to a curve, say $\ell(T)$, when $\varepsilon \to 0$. In fact, it can be proved directly from the system (8) that the ratio $\left(\ell^c_{i_\varepsilon(T)-1} - \ell^c_{i_\varepsilon(T)}\right)/\varepsilon$ tends to a limit say $\dot{\ell}(T)$, solution of the following equation:

$$\sum_{j=1}^{\infty} \left\langle 2\sqrt{G(T)} - j\dot{\ell}(T) \right\rangle^2 = 4\left(1-G(T)\right) \qquad \textit{with} \qquad G(T) = \frac{\overline{V}_m^2}{\left(1-\ell(T)\right)^2}. \tag{12}$$

In eqn (12), $G(T)$ represents the dimensionless potential energy release rate associated to a Griffith debonding until $\ell(T)$ whereas $\dot{\ell} \equiv -d\ell/dT$ represents its growth rate with respect to the rescaled number of cycles. We have thus obtained that the limit fatigue law is a (generalized) Paris

law $\dot{\ell} = f(G)$. Moreover, for *small values* of *G*, this fatigue law is similar to the usual Paris law with the exponent 3/2:

$$\dot{\ell} = \frac{2}{3} G^{3/2} \tag{13}$$

while, for values of *G* near the toughness (*i.e.* near to 1 for dimensionless quantities), we obtain:

$$\dot{\ell} = 2\sqrt{G} - \sqrt{\frac{2}{3}} - \sqrt{1-G}\,. \tag{14}$$

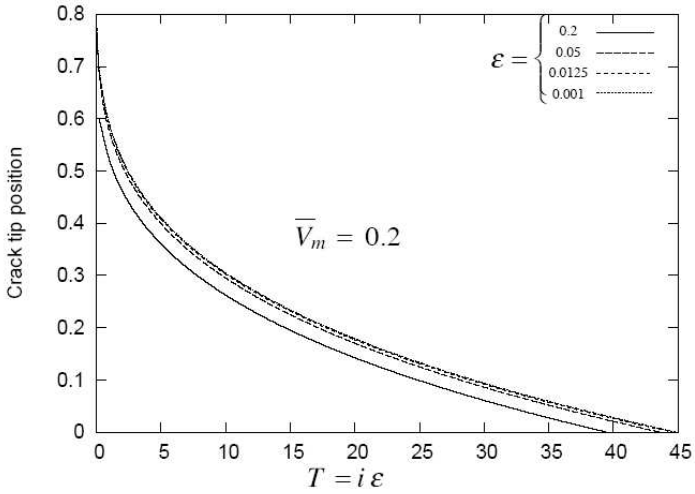

Figure 6. Convergence to a limit fatigue law

Remark 3 : All the results presented here in the simple context of Dugdale surface energy and of perfectible flexible layer can be extended to more general cases. By considering general Barenblatt surface energy or by taking into account the stiffness of the layer, we still obtain fatigue limit laws of Paris type.

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[4] Marigo J.-J. and Truskinovsky L., Initiation and propagation of fracture in the models of Griffith and Barenblatt, *Continuum Mech. Thermodyn.*, 16 (4), 2004, pp. 391-409.