FINITE ELEMENT ANALYSES OF NOTCH-TIP FIELDS OF PLANE PIEZOELECTRIC MATERIALS CONTAINING AN ELLIPTICAL NOTCH[1]

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ABSTRACT

A special finite element based on the complex potential theory and hybrid variational principle is proposed for analyzing mechanical-electrical coupling field of piezoelectric materials containing an elliptical hole. In the formulation, the complex series solutions satisfying the equilibrium equations and compatibility equations are chosen as the displacement and stress fields in the element domain. The series solutions satisfy exactly the D-P condition in advance. While the displacements along the element outer boundaries vary parabolically for an eight-node special element. The element stiffness matrix is then obtained by using the Gauss quadrature method. Numerical examples verify the accuracy and efficiency of the proposed element. It is found that the relationships between the logarithm concentration factor of tangential ($\theta = 0$) stress, electrical displacement, and electrical field at the notch tip and logarithm b/a are linear.

1. INTRODUCTION

PZT materials are widely used as sensors/actuators in smart structural technology due to their fast response and low energy consumption. There are a lot of articles dealing with the damages (cracks) as well as inclusions of PZT material itself by using analytical approaches. Details may be found in a review paper written by Chen and Yu [1]. Since analytical methods can be only used to obtain solutions for simple cases, numerical methods, such as the finite element method, boundary element method, and/or the coupled FEM-BEM, should be resorted for obtaining solutions in general complicated cases. Various conventional finite elements for modeling of piezoelectric materials and smart structures are summarized by Benjeddou [2].

It is costly to model a crack or hole by using ordinary finite elements [3]. While special elements employing hybrid formulations have been succeeded for analyzing crack problems [4-5], and anisotropic plates with an elliptical hole [6-7]. Recently, the special element for ordinary

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materials has been extended to piezoelectric materials for computing edge singularities [8].
In this paper, the special hybrid element with an elliptical hole in reference [6-7] is extended to piezoelectric materials. Complex potential theory and hybrid variational principle [4,5,9] are used to develop the special element. Numerical examples with known analytical solutions are performed to demonstrate the accuracy and efficiency of the proposed elements. It is shown that accurate results around the hole-boundary can be obtained by using only one special element. It is also found that the relationships between the logarithm concentration factor of tangential stress, electrical displacement, and electrical field at the notch tip ($\theta = 0$) and logarithm b/a are fairly linear under simple far-field loadings.

2. REISSNER'S VARIATIONAL PINCIPLE

If the governing differential equations and compatibility equations be satisfied in an element domain. The functional for PZT materials can be simplified as

$$\Pi_R^E = \int_S \tilde{u}_i\sigma_{ij}n_j ds - \frac{1}{2}\int_S u_i\sigma_{ij}n_j ds + \int_S \tilde{\varphi} D_j n_j ds - \frac{1}{2}\int_S \varphi D_j n_j ds - \int_{S_\sigma} u_i \tilde{T}_i ds + \int_{S_\omega} \varphi \tilde{q}_s ds \tag{1}$$

where u_i ,σ_{ij}, φ, D_i are the displacement vector, stress tensor, electric potential, and electric displacement vector, respectively. The quantities with symbol $\sim$ represent the known quantity. And S_σ, S_ω represents the boundary with given tractions, and the normal component of the electric displacement.

It has been proved that the stress, displacement and electrical fields can be expressed by three complex functions, $\phi_k(\xi_k)$ $(k=1,2,3)$, if the complex variable formulation is employed, namely,

$$\sigma_{xx} = 2\,\mathrm{Re}\sum_{k=1}^{3}\mu_k^2\phi_k'(\xi_k)/Z_k'(\xi_k);\ \sigma_{yy} = 2\,\mathrm{Re}\sum_{k=1}^{3}\phi_k'(\xi_k)/Z_k'(\xi_k);$$

$$\sigma_{xy} = -2\,\mathrm{Re}\sum_{k=1}^{3}\mu_k\phi_k'(\xi_k)/Z_k'(\xi_k);$$

$$u = 2\,\mathrm{Re}\sum_{k=1}^{3}p_k\phi_k(\xi_k) + \omega_0 y + u_0; \qquad v = 2\,\mathrm{Re}\sum_{k=1}^{3}q_k\phi_k(\xi_k) - \omega_0 x + v_0$$

$$D_1 = 2\,\mathrm{Re}\sum_{k=1}^{3}\lambda_k\mu_k\phi_k'(\xi_k)/Z_k'(\xi_k);\ \ D_2 = -2\,\mathrm{Re}\sum_{k=1}^{3}\lambda_k\phi_k'(\xi_k)/Z_k'(\xi_k);$$

$$E_1 = 2\,\mathrm{Re}\sum_{k=1}^{3}\left(b_{13}+\delta_{11}\lambda_k\right)\mu_k\Phi_k'(\xi_k)/Z'(\xi_k);$$

$$E_2 = -2\,\mathrm{Re}\sum_{k=1}^{3}\left(b_{21}\mu_k^2 + b_{22} + \delta_{22}\lambda_k\right)\Phi_k'(\xi_k)/Z'(\xi_k)$$

$$\varphi = -2\,\mathrm{Re}\sum_{k=1}^{3}\left(b_{13}+\delta_{11}\lambda_k\right)\mu_k\Phi_k(\xi_k) + \varphi_0 \tag{2}$$

In Eq. (2), the symbol *Re* represents the real part of a complex function and

$$Z_k^{'}(\xi_k) = \frac{dZ_k}{d\xi_k} = \frac{a - i\mu_k b}{2} - \frac{a + i\mu_k b}{2}\xi_k^{-2} \tag{3}$$

where a and b are the semi-axes of the elliptical hole, and z_k are defined as

$$z_k = x + \mu_k y \tag{4}$$

The complex variable μ_k (k=1,2,3) is the roots of the following equation:

$$\begin{aligned} &a_{11}\delta_{11}\mu^6 + \left(a_{11}\delta_{22} + 2a_{12}\delta_{11} + a_{33}\delta_{11} + b_{21}^2 + b_{13}^2 + 2b_{21}b_{13}\right)\mu^4 \\ &+\left(a_{22}\delta_{11} + 2a_{12}\delta_{22} + a_{33}\delta_{22} + 2b_{21}b_{22} + 2b_{22}b_{13}\right)\mu^2 + \left(a_{22}\delta_{22} + b_{22}^2\right) = 0 \end{aligned} \tag{5}$$

The explanation for undefined symbols and detailed derivations may be found in [10-12]. Generally speaking, it is impossible to find closed form solutions for $\phi_k(\xi_k)$ for arbitrary boundary conditions. Therefore, a finite series formulation is adopted herein in developing the special element, namely,

$$\phi_1(\xi_1) = \sum_{j=-N}^{M} A_j \xi_1^j, \quad \phi_2(\xi_2) = \sum_{j=-N}^{M} B_j \xi_2^j, \quad \phi_3(\xi_3) = \sum_{j=-N}^{M} C_j \xi_3^j \tag{6}$$

where A_j, B_j, C_j are complex coefficients defined by

$$A_k = A_{xk} + iA_{yk}, \; B_k = B_{xk} + iB_{yk}, \; C_k = C_{xk} + iC_{yk}, \quad i = \sqrt{-1} \tag{7}$$

Without loss of the generality, set M equal N. Since terms with $j = 0$ in Eq. (6) contribute no stresses and electric displacements, thus are discarded in the formulations.

For the D-P condition (traction-free and charge-free) along the cavity boundary, one has

$$\mathrm{Re}\sum_{k=1}^{3} \phi_k(\xi_k) = 0, \quad \mathrm{Re}\sum_{k=1}^{3} \mu_k\phi_k(\xi_k) = 0, \quad \mathrm{Re}\sum_{k=1}^{3} \lambda_k\phi_k(\xi_k) = 0 \tag{8}$$

Equation (8) is to be used to satisfy the boundary conditions on the elliptical hole boundary in advance, thus, the total number of independent coefficients in Eq. (6) is reduced to half. Using Eqs. (2), (3), (6), and (8), the displacement, stress, and electrical fields in the element domain, u_i, σ_{ij}, φ, D_i, and traction $\mathbf{t}$ on the element boundary can be symbolically expressed as

$$\sigma = S\beta, \qquad u = U\beta, \qquad \mathbf{t} = \mathbf{N}\boldsymbol{\sigma} = \mathbf{NS}\boldsymbol{\beta} \tag{9}$$

where vector β contains only the independent variables since Eq. (8) has been used. For example, if $N = M = 4$, one has

$$\beta^T = \left\{A_{x1}, A_{y1}, ..., A_{x4}, A_{y4}, B_{x1}, B_{y1}, ..., B_{x4}, B_{y4}, C_{x1}, C_{y1}, ..., C_{x4}, C_{y4}\right\} \tag{10}$$

The external boundary displacement vector and electric potential, $\tilde{\mathbf{u}}$, is independently assumed in terms of the nodal displacements and electric potentials, namely,

$$\tilde{\mathbf{u}} = \mathbf{L}\boldsymbol{\delta} \tag{11}$$

where elements of matrix $\mathbf{L}$ are the interpolation functions defined only along the element outer boundary, and elements of vector $\boldsymbol{\delta}$ are the nodal displacement and electric potential. To be jointed with special or conventional elements, the interpolation functions for the special element developed herein are chosen to be displacement shape functions of the adjacent conventional elements.

Following the common procedures in [4-5], the stiffness matrix is derived as

$$\mathbf{K}=(\mathbf{G}^{\mathrm{T}}\mathbf{H}^{-1}\mathbf{G}) \tag{12}$$

where

$$\mathbf{G} = \int_{S} (\mathbf{NS})^{\mathrm{T}}\,\mathbf{L}ds, \qquad \mathrm{H} = \frac{1}{2}\int_{S} [(\mathbf{NS})^{\mathrm{T}}\,\mathbf{U} + \mathbf{U}^{\mathrm{T}}\,(\mathbf{NS})]ds \tag{13}$$

Detailed formulations may be referred to reference [12].

3. EIGHT-NODE SPECIAL ELEMENT

Various special elements containing an elliptical hole with differential nodes can be formulated. As an example, an eight-node special element, schematically shown in Fig. 1, is considered. Each node has three degrees of freedom (DOFs), e.g., u_1, v_1, φ_1 at node 1. The poling direction is assumed in the y-direction. The lengths of the semi axes of the hole are a and b and $N = M = 4$. It can be seen that there are totally 24 unknowns in the $\boldsymbol{\beta}$ matrix and the number of unknowns is equal to the number of the degrees of freedoms of the element, satisfying the requirement for formulations of hybrid elements.

The shape functions are chosen in parabolic forms, namely,

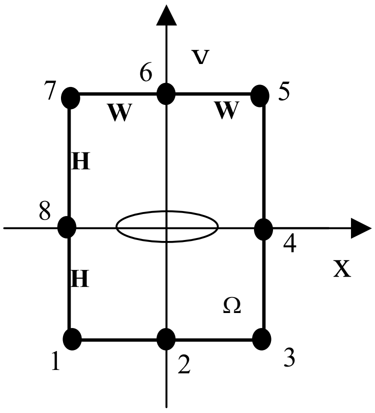
$$\mathrm{N}_1 = (s-1)(2s-1);\ \ \mathrm{N}_2 = 4s(1-s);\ \ \mathrm{N}_3 = s(2s-1) \quad s \in [0,1] \tag{14}$$

where $s = S / L_{ij}$, S is the arc variable along the element outer side, and L_{ij} is the length of the element side ij. It should be pointed out that linear forms could also be used.

4. NUMERICAL EXAMPLES

A computer program is written. Various examples have been studied to test the proposed element. The material is assumed a PZT-4 ceramic and the reduced material constants in [10] are used in the analyses. Set H=W=50mm and $a / H = 0.02$ to simulate the infinite piezoelectric medium. Due to space limitations, only a few examples and results obtained by using one special element are listed below.

Example 1 Consider an infinite piezoelectric medium with a circular hole ($a=b$) subjected to far field electrical loading, $D_2^{\infty} = D_0$. The finite element results for the electric fields of E_r and E_θ normalized with respect to D_0 are shown in Fig. 2 (symbols). It can be seen that numerical data agree well with analytical results (solid lines) [10, 3]. To achieve the same accuracy, 960 eight-node ordinary parametric elements are needed to model a quarter of the plates [3]. It is obvious that the computational efficiency is high for the proposed finite element in this case. Since the analytical data in [10] is incorrect, the data in Fig. 2 are recalculated.



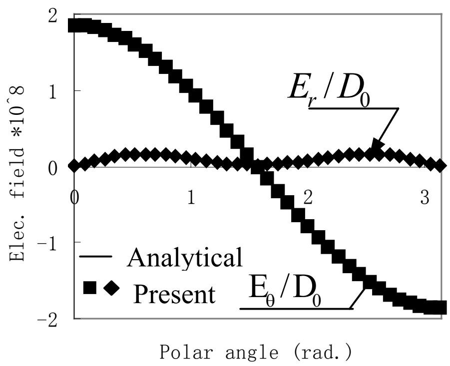
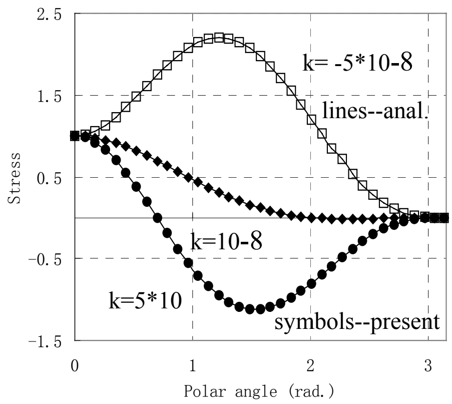


Fig. 1 An eight-node special element

Fig. 2 Variations of E_r and E_θ with θ

Example 2 Consider an infinite piezoelectric medium with a center crack of length $2a$ ($b=0$) under combined mechanical and electrical loadings, namely, $\sigma_2^\infty = \sigma_0$ and $D_2^\infty = D_0$. To present the results and compare with analytical solutions, the polar coordinate system with the origin at the crack-tip is introduced. Define $k = D_2^\infty / \sigma_2^\infty$ and $\sigma = \frac{\sigma_\theta \sqrt{2r}}{\sigma_2^\infty \sqrt{a}}$, where σ_θ is the stress computed at $r = 6\times 10^{-4}\, mm$ in θ direction. The finite element results for stress σ are shown in Fig. 3 (symbols) with three different values of k (5×10^{-8}, 10^{-8}, -5×10^{-8}). It can be seen that all numerical results are well compared with the analytical solutions, recalculated herein due to the incorrectness of the numerical data present in [11].



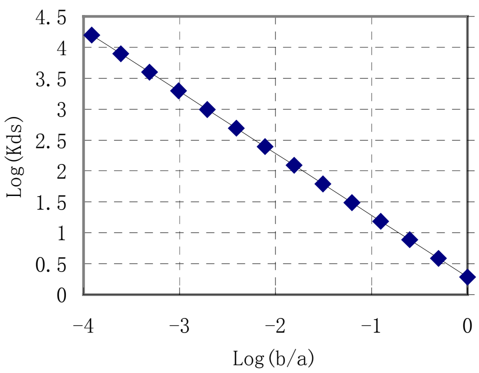


Fig. 3 Stress distributions around the crack tip under combined loadings ($k = D_2^\infty / \sigma_2^\infty$)

Fig. 4 Double Logarithm plot of Kds with b/a

Example 3 Consider piezoelectric rectangular plates with a central elliptical cavity. The plate is subjected to either far field mechanical loading σ_0 or electric loading D_0. To investigate the concentration of stress and electrical quantities in the tangential direction at $\theta = 0$ with varying b/a ratios, set $a = 1$ and change b so that the b/a ratio ranging from 1 to 0.0001. In other words, a circular hole approaches a crack gradually. Figure 4 shows the finite element data of Kds (symbols) for the case of far field mechanical loading. Define the concentration factors of tangential electrical displacement D_θ at $x = a, y = 0,$ and $\theta = 0$, namely, $Kds = D_{\theta=0} \times 10^{10} / \sigma_0$. It is found that the relationships between logarithm Kds and the logarithm of ratio b/a are linear for all three factors, described by

$$\log(Kds) = 0.2847263 - 1.00085 \times \log(b/a) \tag{16}$$

Similar linear relationships are also found for other mechanical and electrical quantities and loadings.

5. CONCLUSIONS

Special element containing an elliptical hole is developed for stress and electric field analysis of piezoelectric plates with defects. Numerical examples are performed to demonstrate the efficiency and accuracy of the proposed element. For finite plates containing an elliptical hole under mechanical or electrical loadings, it is found that the relationship between the logarithm concentration factor of tangential stress, electrical displacement, and electrical field at $x=a, y=0$ and logarithm b/a ratios is fairly linear.

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