CRACK-LIKE TRANSITION WAVE IN LATTICES

L.I. SLEPYAN
Department of Solid Mechanics, Materials and Systems, Tel Aviv University
Ramat Aviv 69978 Israel

ABSTRACT

Steady-state dynamic problem is studied for discrete square and triangular lattices whose massless bonds follow a trimeric piecewise linear force–elongation diagram. In a general case, a prestressed or an active lattice is considered. The transition wave is found analytically as that localized between two neighboring lines of the lattice particles. The transition wave itself is accompanied by dissipation waves carrying energy away from the transition front and, in general, away from the transition line. However, in the case of the transition leading to an increased tangent modulus of the bond, there exist non-divergent dissipative waves exponentially localized in a vicinity of the transition line behind the transition front. In particular, where the second branch has zero resistance, the transition wave becomes a crack. In this case, the solution for an infinite lattice includes an energy flux from a remote source, and no prestress is assumed. The Green function corresponding to a general value of the second branch tangent modulus is expressed via the known lattice-with-a-moving-crack fundamental solutions. Mathematically, this allows to study the transition wave in two-dimensional lattices as a one-dimensional problem. The analytical results are compared to the ones obtained for a continuous elastic model and for the lattice-related version of one-dimensional Frenkel-Kontorova model.

1 INTRODUCTION

The discrete lattice is considered consisting of particles connected by massless bonds. The bonds follow the force–elongation diagram presented in Fig. 1. If such a lattice is initially stressed, it can behave as an active structure. As a result of a disturbance, it can release initially stored energy for the dynamic transition. In particular, the transition can arise as a localized wave propagating between two neighboring lines of the lattice particles. This wave results in the unloading of the outer bonds which thus remain in the initial phase. Note, however, that this localized transition wave

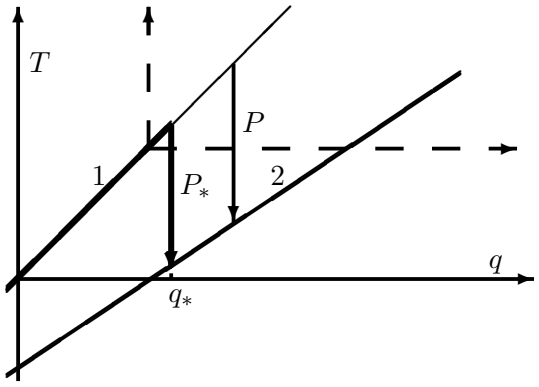

Fig. 1. The piecewise linear force–elongation diagram: 1 The first, initial branch, $T = q$. 2 The second branch, $T = q - P_* - (1 - \gamma)(q - q_*)$; it comes in force when the elongation reaches the critical value, $q = q_*$. The vertical distance between the branches is denoted by P. The case of a crack corresponds to $P_* = q_*, \gamma = 0$. The dotted axes correspond to the diagram for an initially stressed lattice. This phase transition is assumed irreversible.

is accompanied by the structure-associated sinusoidal waves excited by the transition front. These waves can carry energy away from the transition line. Such a line transition wave looks like a propagating crack bridged by fibres as in fracture of fibre-reinforced composites or in polymers or fabric crazing.

The Fourier transform and the Wiener–Hopf technique are used allowing the detailed solution to a two-dimensional steady-state problem to be obtained including that for the structure of the radiation (dissipation) during the moving transition. This becomes possible for discrete square and triangular lattices with bistable bonds characterized by the irreversible stress-strain dependence, Fig. 1.

For our goal it is convenient to reformulate this nonlinear problem as follows. Consider the linear lattice all the bonds of which are in the initial phase for any q. Introduce self-equilibrated pairs of external forces, $P(\eta), \eta = x - vt$, applied to the masses connected by the transition-line bonds. The forces are directed along the corresponding bond; they must compensate the difference in the tensile forces between the real and the initial-phase dependencies. In general, these forces depend on the elongation. The intact bonds together with the external forces act on the transition-line masses, $n = 0$ and $n = -1$, in the same way as if the bonds follow the given nonlinearity. Hence, such a reformulation does not influence the lattice dynamic behavior. The elongation caused by these forces is expressed in terms of the corresponding fundamental solution, $Q(\eta)$, which reflects the structure of the intact lattice as a whole. In turn, this function is expressed via the known lattice-with-a-moving-crack fundamental solutions, and the problem for the two-dimensional lattices is reduced to a one-dimensional one. Non-dimensional variables are used.

2 GOVERNING EQUATION

For an initially stressed lattice it is assumed that no additional external sources exist, and the phase transition wave propagates with a constant speed, v, using the energy released due to the transition. In this case, the above-mentioned compensating forces gives us a nontrivial solution. In terms of the Fourier transform the additional elongation of the transition-line bonds, that is, the elongation relatively the dotted axes, can thus be represented as

$$q^F(k) = Q^F(0 + \mathrm{i}kv, k)P^F(k)\,, \tag{1}$$

where the fundamental solution, $Q(\eta)$ corresponds to impulsive forces as $P = \delta(\eta)$. Further, for the diagram in Fig. 1 the forces, $P^F(k)$, are

$$P^F = P_*(0 + \mathrm{i}k)^{-1} + (1 - \gamma)\,\left[q_- - q_*(0 + \mathrm{i}k)^{-1}\right]\,, \tag{2}$$

where the subscripts '$\pm$' are used for the right/left sided Fourier transforms ($q_+ + q_- = q^F$) and γ is the second branch nondimensional tangent modulus (the first branch modulus is equal to one). Here $q(0) = q_*$ (for a crack at $\eta < 0$: $P_* = q_*, \gamma = 0$). Eqn (1) becomes

$$q_+ + \left[1 - (1 - \gamma)\,Q^F\right]\,q_- = P_{**}\frac{Q^F}{0 + \mathrm{i}k}\,, \quad P_{**} = P_* - (1 - \gamma)\,q_*\,. \tag{3}$$

For $\gamma = 0$ the left-hand side of this equation must coincide with that for the crack problem. It follows that $Q^F = 1 - 1/L$, where $L(0 + \mathrm{i}kv, k)$ is the crack related Green function (see Slepyan [1]). For $\gamma \neq 1$ we get the governing equation in the form

$$q_+ + q_-/\mathcal{L} = \frac{q^0}{0 + \mathrm{i}k}(1 - 1/\mathcal{L})\,, \quad \mathcal{L} = \frac{L}{\gamma L + 1 - \gamma}\,, \quad q^0 = \frac{P_*}{1 - \gamma} - q_*\,, \tag{4}$$

while for $\gamma = 1$ the problem has a straightforward solution

$$q^F = P_* Q^F/(0 + \mathrm{i}k)\,. \tag{5}$$

The following properties of the function $\mathcal{L}$ are important. For any lattice

$$\begin{gathered}\mathrm{Ind}\mathcal{L} = 0\,, \quad \mathcal{L} \to 1 \quad (k \to \pm\infty)\,, \quad \mathcal{L} \to 1/\gamma \quad (k \to 0)\,, \\ (-\infty, 0] \not\subset \mathcal{L}(s + \mathrm{i}kv, k)\,, \quad -\pi < \arg\left[k\mathcal{L}(s + \mathrm{i}kv, k)\right] \leq 0 \quad (s > 0)\,.\end{gathered} \tag{6}$$

Further, this function has singular points of the square-root type (the number of which depends on the speed) and, if $\gamma > 1$, it has poles, including twofold ones for some regions of the speed, and threefold ones for some combinations of v and γ. Physically, these poles reflect the fact that, in the discrete lattices with a stiffer layer, there exist non-divergent sinusoidal waves exponentially localized in a vicinity of this layer. These waves, as well as some waves associated with other singular points, appear in the solution to eqn (4) as the dissipative waves.

3 SOLUTION

The solution for $\gamma = 1$ is presented in eqn (5), and we now consider the case $\gamma \neq 1$. The function $\mathcal{L}$ can be factorized as $\mathcal{L} = \mathcal{L}_+\mathcal{L}_-$ using the Cauchy-type integral. If $s > 0$ the function $\mathcal{L}_+(s + \mathrm{i}kv, k)$ $[\mathcal{L}_-(s + \mathrm{i}kv, k)]$ has no zeros and singular points in the upper (lower) half-plane with the real axis. Further

$$\mathcal{L}_+ \to 1 \quad (k \to \mathrm{i}\infty)\,, \quad \mathcal{L}_- \to 1 \quad (k \to -\mathrm{i}\infty)\,. \tag{7}$$

Noting that $\mathcal{L}(0) \equiv \mathcal{L}(+0, 0) = 1/\gamma$ we now can rearrange eqn (4) for $\gamma > 0$ ($\gamma \neq 1$) in the form

$$\begin{aligned} \mathcal{L}_+ q_+ + q_-/\mathcal{L}_- = q^0[\mathcal{L}_+ - 1/\mathcal{L}_-]/(0 + \mathrm{i}k) = C_+(k) + C_-(k), \\ C_+(k) = q^0[\mathcal{L}_+ - \mathcal{L}_+(0)]/(0 + \mathrm{i}k) = q^0[\mathcal{L}_+(0) - \mathcal{L}_+]/(0 - \mathrm{i}k)\,, \\ C_-(k) = q^0[\mathcal{L}_+(0) - 1/\mathcal{L}_-]/(0 + \mathrm{i}k)\,, \\ \mathcal{L}_\pm(0) = \frac{1}{\sqrt{\gamma}}\mathcal{R}^{\pm 1}, \quad \mathcal{R} = \exp\left[\frac{1}{\pi}\int_0^\infty \frac{\operatorname{Arg}\mathcal{L}(\xi)}{\xi}\,\mathrm{d}\xi\right]. \end{aligned} \tag{8}$$

The functions $C_+(k)$ and $C_-(k)$ have no singular points in the upper and lower half-planes, respectively [point $k = 0$ is regular for $C_+(k)$; this allows us to change $0 + \mathrm{i}k$ to $-(0 - \mathrm{i}k)$]. Now, following the Wiener-Hopf technique and taking into account the condition $q = 0$ at $\eta = +\infty$, we can simply separate the corresponding functions in eqn (8) and find the solution as

$$q_+ = C_+(k)/\mathcal{L}_+\,, \quad q_- = C_-(k)\mathcal{L}_-\,. \tag{9}$$

For the genuine crack problem $P_* = q_*, \gamma = 0, q^0 = 0, \mathcal{L} = L$, and the governing equation (4) becomes homogeneous

$$q_+ + q_-/L = 0\,. \tag{10}$$

In this case, however, $\mathcal{L} = L = L_+L_-, L_\pm \sim f_\pm(v)/\sqrt{0 \mp \mathrm{i}k}$ $\quad (k \to 0)$ (these relations are valid for subcritical speeds), and there exist a homogeneous solution as

$$q_+ = A(v)[(0 - \mathrm{i}k)L_+]^{-1}\,, \quad q_- = A(v)L_-/(0 + \mathrm{i}k)\,, \tag{11}$$

where the function $A(v)$ is defined by the 'macrolevel' energy release rate. In particular, from this it follows that

$$q(0) = q_* = q^0\left(\frac{1}{\sqrt{\gamma}}\mathcal{R} - 1\right). \tag{12}$$

Physically, the difference between the crack problem, $\gamma = 0$, and the transition problem, $\gamma > 0$, is the following. In the former there is no prestress ($P_* = q_*$; otherwise, the steady-state solution does not exist), and there is the energy flux from infinity, while in the latter the elongation is finite $[q(-\infty) = 0]$, and there is no energy flax from infinity. In this latter case, P_* must exceed q_* to provide a positive energy release (in this connection, see next section).

4 ENERGY RELEASE AND THE TRANSITION WAVE SPEED

The phase transformation path on the 'macrolevel', that is, in the long-wave approximation where no high-frequency dissipative waves are detected, corresponds to the horizontal segment along the dotted q-axis between the first and the second branches of the diagram in Fig. 1. The first (small) triangle formed by the diagram and this axis represents the energy barrier, while the second one defines the released energy. The resulting energy release per bond is the difference between the areas of the latter and the former; it is

$$G = \frac{1}{2\gamma}(P_* - q_*)^2 - \frac{1}{2}q_*^2 \,. \tag{13}$$

In accordance with the obtained solution, this energy is radiated away from the transition front with sinusoidal dissipative waves.

The energy dissipation per unit time is vG for the square-cell lattice and $2vG$ for the triangular-cell lattice. In terms of the resulting energy release,

$$q_* = \frac{P_*}{\sqrt{\gamma}\sqrt{1+2G^0}+1} \quad \left(G^0 = \frac{G}{q_*^2}\right) \tag{14}$$

and Eq. (12) can be rewritten in the form

$$\mathcal{R} = \frac{1+\sqrt{\gamma}\sqrt{1+2G^0}}{\sqrt{\gamma}+\sqrt{1+2G^0}} \,. \tag{15}$$

This equation defines the transition wave speed as a function of G^0 [$\mathcal{R}$ is defined in (8) as a function of v].

In addition to the condition $q(0) = q_*$, the analytical solution must be admissible in the sense that the bond elongation in the initial phase must be below the critical value at $\eta > 0$. This condition also concerns all the bonds outside the transition line. Otherwise, the considered single-line steady-state transition wave does not exist. This condition is similar to that in fracture of lattices (Marder and Gross [2]). In this connection, consider a necessary condition as $\mathrm{d}q(\eta)/\,\mathrm{d}\eta \le 0 \quad (\eta = 0)$.

The solution allows to find the following expression for this derivative

$$\frac{\mathrm{d}q(\eta)}{\mathrm{d}\eta} = \frac{\mathcal{R}}{\pi\sqrt{\gamma}}\left(\frac{P_*}{\gamma-1}+q_*\right)\int_0^\infty \ln|\mathcal{L}(\xi)|\;\mathrm{d}\xi \quad (\eta = 0)\,. \tag{16}$$

This result can be directly used if $\gamma \neq 1$, while the expression for $\gamma = 1$ can be obtained as the limit, $\gamma \to 1$; it is

$$\frac{\mathrm{d}q(\eta)}{\mathrm{d}\eta} = -\frac{P_*}{\pi}\int_0^\infty \left[1 - \Re\frac{1}{L(k)}\right]\,\mathrm{d}k\,, \tag{17}$$

The function $G^0(v)$ reaches its minimum roughly at $v = 0.5$. After this, G^0 monotonically grows as v grows. This branch corresponds to admissible speeds, but the left branch does not.

5 SOME OTHER MODELS

The related steady-state problem for a continuous elastic medium can be formulated as that for the upper half-plane, $y > 0$, with the following boundary conditions at $y = 0$ (it is written here for mode I):

$$\sigma_{xy} = 0 \quad (-\infty < \eta < \infty), \quad \sigma_{yy} = -P_{**} + \kappa u_y \quad (\eta < 0), \quad u_y = 0 \quad (\eta > 0)\,, \quad (18)$$

This formulation is the same as that for a crack at $\eta < 0$ whose faces are loaded by the above-expressed traction. So, the traction corresponds to the crack faces' interaction in the second phase of the initially stressed material — similarly to that prescribed by the diagram in Fig. 1. Also, the energy criterion for the transition is used as $G = G_c$ at $\eta = 0$. This formulation leads to an explicit solution (in terms of the Fourier transform), and the total energy release rate appears as $G = P_{**}^2/\kappa$. Thus, this solution exists if the required energy, G_c, is exactly equal to the energy, G, produced by the traction. If the prestress is greater than the critical one, that is, if $G > G_c$, the steady-state regime does not exist, and the speed is not a constant, but asymptotically verges towards the critical value. Thus the classical, homogeneous-body formulation leads to a very different result in comparison with that based on the lattice model where the steady-state regime exists for any, but not too small, resulting energy release rate, and the speed is energy-release-rate-dependent.

Further, the comparison with the results following from the related version of one-dimensional Frenkel-Kontorova model (Frenkel and Kontorova [3]; Braun and Kivshar [4]) shows that qualitatively the G^0 – v dependencies for FK model are similar to those for the lattice. Quantitatively they are also not too far from each other. The basic differences are in the structure of the dissipative waves. The comparison thus evidences that the transition wave speed not too strongly depends on directions of the radiation; it mainly depends on the energy release rate and γ.

In conclusion, note that a detailed examination of the titled problem including some numerical simulations are presented in Slepyan and Ayzenberg-Stepanenko [5].

REFERENCES

1 Slepyan, L.I. Models and Phenomena in Fracture Mechanics. Springer, 2002.

2 Marder, M., Gross, S. Origin of crack tip instabilities. J. Mech. Phys. Solids 43, 1-48, 1995.

3 Frenkel, J. and Kontorova, T. On the theory of plastic deformation and twinning. Sov. Phys. JETP 13, 1-10, 1938.

4 Braun, O. and Kivshar, Y. Nonlinear dynamics of a Frenkel-Kontorova dislocation in a one-dimensional crystal. Phys. Rev., V138(3A), A763-A766, 1998.

5 Slepyan, L.I. and Ayzenberg-Stepanenko, M.V. Localized transition waves in bistable-bond lattices. J. Mech. Phys. Solids 52, 1447-1479, 2004.