Dislocation model of an asymmetric weak zone for problems of interaction between crack-like defects

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Crack-like defect models range from the well-known traction free Griffith-Irwin cracks, Barenblatt cuts containing small process zones at tips, to cuts subjected to cohesive forces over their entire length. This last type of defect may be regarded as a weak zone (WZ) in the solid which is normally closed but which can open progressively under sufficiently large remote external tensile forces. The fact that the WZ begins to open only when the remote force has reached a certain threshold level distinguishes the WZ from a real cohesive crack. The adhesive forces can be of very different physical origin - atomic, dislocational, localized porosity, etc. Healed cracks in glaciers and in the earth's crust are also WZs. The length of WZs can thus range from a few nanometers to hundreds of kilometers. It is interesting to investigate the fundamental behaviour of the WZs during their interaction with other defects. Note that the direct approach using some cohesive force-opening relationship leads to a complicated problem which is reduced to the solution of a system of nonlinear singular integral equations in [1]. Another approach was suggested in [2], whereby the cohesive forces were expressed in a series containing N terms with N free parameters. The free parameters were determined by imposing physically consistent conditions on the solution.

This paper focuses on two topics. Firstly, guided by the results obtained in [2], a general approach to examining the behaviour of a WZ embedded in a very asymmetric external tensile stress field is developed by prescribing *a priori* the WZ asymmetric opening by a two-parameter basis function which meets all physical constraints. Secondly, this approach is exemplified on the problem of interaction between a long interface crack subjected to wedge opening forces and a short collinear WZ. The latter is separated from the main crack by a small strong microstructural feature of the material (i.e. by a small obstacle). The key questions that will be addressed are: (i) when does the WZ become the nucleus of a cohesive crack on its own without linking with the pre-existing long crack, and (ii) when does the WZ force the obstacle to rupture allowing the pre-existing crack to link with it. The critical applied load levels corresponding to these limiting situations will be determined.

1. An asymmetric weak zone modelled as a special dislocation

In contrast to the direct method used in [2], we shall take an inverse approach when the WZ is located in an asymmetric tensile stress field and search for a basis function of asymmetric opening displacement of WZ in the form

$$w = w_0 T^{3/2}, \qquad T = tG(X,\ \eta), \qquad |\eta| < 1 \tag{1}$$

The function $G(X,\ \eta)$ depends on the parameter of asymmetry, η, i.e. the distance by which the maximum of $w(X)$ is displaced from the centre of WZ, $X = 0$. It is assumed that $\eta > 0$, if the maximum is displaced to the left of centre.

We shall seek the unknown function $G(X,\ \eta)$ under the following mathematical constraints which result from obvious physical and geometrical considerations: (i) it is even with respect to a simultaneous change of sign of both variables: $G(X,\ \eta) = G(-X,\ -\eta)$; (ii) it is normalized such that : $T = (-\eta,\ \eta) = 1 \iff G(-\eta,\ \eta) = (1-\eta^2)^{-1}; G(X,\ 0) = 1$; (iii) it has a maximum at point $X = -\eta$: $\partial T/\partial X(-\eta,\ \eta) = 0$; (iv) it is bounded at the right tip $X = 1$ as $\eta \to -1+0$, and (v) $0 < G(1,\ \eta) < G(-1,\ \eta)$, when $\eta > 0$.

We shall search for a rational function of parameter η as a possible form of $G(X,\ \eta)$, taking into account immediately the first of the five constraints above

$$G(X, \eta) = \sum_0^\infty \frac{A_n \eta^{2n}}{\eta X B_n \eta^{2n} + C_n \eta^{2n}}$$

By subjecting this rational function to the remaining four conditions, we obtain first a discrete recursive system of equations for the coefficients $A_n,\ B_n$, and C_n and then, rather unexpectedly, the following simple result:

$$T(X, \eta) = \frac{1 - X^2}{(1+\eta^2)(1+\alpha X)}, \qquad \alpha = \frac{2\eta}{1+\eta^2}. \tag{2}$$

$$w_0 = w_* \iff \sigma(-\eta) = 0. \tag{3}$$

When $\eta \neq 0$, the asymptotic behaviour of (2) over a "large" neighbourhood of the WZ tips resembles the expected cusp. In fact, the condition (ii) above was chosen with this requirement in mind.

By prescribing the displacement jump in the form (1) we are actually considering a kind of special dislocation. The parameter η is determined from the additional condition that the minimum of the cohesive force occurs at the location of the maximum of the opening displacement $w(X)$, i.e. at $X = -\eta$. We also require that $\sigma(X)$ and $w(X)$ be positive right up to the instant when the maximum opening displacement of WZ reaches the critical value w_* – a material constant. The critical value w_* establishes a limit on the external applied stress σ_*.

To complete the statement of the model defect system it is necessary to stipulate another material constant, namely the threshold value of the nominal (i.e. when WZ is absent) external applied stress σ_∞, below which the WZ cannot open, σ_{th}. In practice, we only consider situations where $\sigma_{th} < \sigma_\infty < \sigma_*$. In this range, w_0 in (3) remains a free parameter as a consequence of a certain indeterminacy inherent to the present model.

2. Interaction between a long interface crack and a collinear short weak zone

The phenomenon of intermittent crack growth has often been observed in structured materials, e.g. in toughened ceramics [3]. In all likelihood, Broberg [4] was the first to propose a possible explanation for this phenomenon. The crack front periodically meets a microstructural feature (an obstacle) of enhanced strength and is arrested by it. Ahead of this obstacle, in the region of normal strength, micro-defects are formed in time. These alter the local stress field and cause a stress concentration on the obstacle which ruptures as a result, allowing the crack front to advance until it meets another obstacle. Many such situations have been reported in the literature [5]. It is therefore important to investigate how a crack overcomes an obstacle when there is a weak zone (WZ) ahead of the obstacle.

2.1 Problem formulation and general solution

Consider the plane elastic problem of a semi-infinite crack with a small cohesive zone at its tip and a WZ along the interface between two dissimilar linear isotropic half-planes (Figure 1). The crack faces are subjected to wedge forces at unit distance $x=-1$ from the tip. The cohesive forces over the small process zone near the tip of the crack, $-\ell < x < 0$, are distributed quadratically [2]: $\sigma_c = \sigma(0)(1-R^2), R = x/\ell$.

The unit distance from the crack tip to the points of application of the wedge forces sets the largest length scale of the problem. All other distances are much smaller than unity. In the interval $a < x < b$, unknown cohesive forces prevent the opening $w(x)$ of the WZ. The latter is prescribed through the special dislocation (1) and (2) with the following co-ordinate transformation (where L is the half-length of WZ)

$$X = \frac{x}{L} - D, \qquad D = \frac{b+a}{b-a} > 1, \qquad L = \frac{b-a}{2}.$$

The stress fields, σ^j_{km}, and derivatives the displacement vector, $\mathbf{u} = (u, v)$, with respect to x can be expressed through the complex potentials, $\chi_j(z),\ z = x + iy$, [6], subject to the following conditions of continuation across the interface

$$\chi_1(z) = \overline{-\chi_1(\overline{z})}, \qquad \chi_2(z) = \overline{\chi_2(\overline{z})}, \qquad \Im z > 0. \tag{4}$$

The potentials $\chi_j(z)$ $(j = 1, 2)$ are linear transforms of the well-known Kolosov-Muskhelishvili complex potentials [6]. They are holomorphic in the whole z-plane except along the axis $y = 0$ and almost everywhere have finite limits as $y \to \pm 0$. At $y = \pm 0$, the normal and shear stresses and the derivatives of the displacement vector with respect to x are expressed as follows

$$\begin{gathered}
\sigma_{12} = \Im\chi_1, \qquad \sigma_{22} = \Re\chi_2 \\
U^{\pm} = -\Re\{b_j\chi_1^{\pm} + a_j\chi_2^{\pm}\}, \qquad V^{\pm} = \Im\{a_j\chi_1^{\pm} + b_j\chi_2^{\pm}\}, \\
s(x) = [U] = U^{+} - U^{-} = -q\Re\{\chi_1 + \beta\chi_2\}, \qquad w(x) = [V] = q\Im\{\beta\chi_1 + \chi_2\}, \\
\mathbf{U} = \mathrm{d}\mathbf{u}/\mathrm{d}x, \qquad q = b_1 + b_2, \qquad \beta = (a_1 - a_2)/q, \\
4\mu_j a_j = 1 - \kappa_j, \qquad 4\mu_j b_j = 1 + \kappa_j, \qquad j = 1, 2
\end{gathered} \tag{5}$$

where $\kappa_j = 3 - 4\nu_j$ in plane strain and $\kappa_j = (3-\nu_j)(1+\nu_j)^{-1}$ in generalized plane stress. $w(x)$ and $s(x)$ are the normal and horizontal displacement discontinuities and β is Dundurs' mismatch parameter.

The solution of this generalised Riemann-Hilbert problem for the two complex potentials [6] contains rapidly oscillating cofactors. But as shown in [7], they influence only a fairly small (less than 10^{-4}) zone around the crack tips. Besides, relatively minor shear stresses appear along the sections with full contact. These effects do not play an important role in the problem at hand and can be neglected for simplicity. The coupled boundary-value problem is then uncoupled with $\chi_1(z) \equiv 0$.

With these approximations which are the better the smaller the value of β, the normal stresses are

$$\sigma_1(x) = \frac{1}{\pi q}\int_a^b \sqrt{\frac{|\xi|}{|x|}}\frac{w'(\xi)\mathrm{d}\xi}{\xi - x}, \qquad \sigma_2(x) = \frac{1}{\pi}\int_{-\infty}^0 \sqrt{\frac{|\xi|}{|x|}}\frac{\sigma_0(\xi)\mathrm{d}\xi}{x-\xi}, \tag{6}$$

where, $\sigma_1(x)$ is the contribution of WZ, $\sigma_2(x)$ is the nominal stress in the absence of WZ, and $\sigma(0)$ is calculated from the finiteness condition. The normal stress $\sigma_2(x)$ varies rapidly over the WZ, if the zone $a < x < b$ is situated close to the long crack tip $x = 0$. The stress field in the vicinity of the long crack is however only altered significantly by the presence of the WZ when the latter is very close to it ($a \ll b$).

Let us now calculate the necessary expressions starting from (1) and (2)

$$w'(x) = \frac{\mathrm{d}w}{\mathrm{d}X}\frac{\mathrm{d}X}{\mathrm{d}x} = -\frac{3w_0}{L}g(X), \qquad a < x < b,$$

$$g(X) = \frac{(1+\eta X)(\eta + X)\sqrt{1-X^2}}{(1+\eta^2+2\eta X)^{5/2}}, \qquad X = \frac{x}{L} - D.$$

Substituting this expression into the integral for $\sigma_1(x)$ in (6), we get

$$\sigma_1(x) = \frac{3W_0}{\pi L}\int_{-1}^{1}\sqrt{\frac{\xi + D}{X + D}}\frac{g(\xi)\mathrm{d}\xi}{\xi - X}, \qquad W_0 = \frac{w_0}{q\Sigma} \tag{7}$$

Likewise, the stress component $\sigma_2(x)$ is given by

$$\sigma_2(x) = \frac{(1+I)S(R) - I - \frac{x}{1+x}}{\pi\sqrt{x}}, \quad I = W_0 I_0 = W_0\int_{-1}^{1}\left(\frac{T(\xi)}{\xi + D}\right)^{3/2}\frac{\mathrm{d}\xi}{2\sqrt{L}}, \tag{8}$$

$$S(R) = \frac{5}{4}\left\{\sqrt{R}(1-R^2)\mathrm{arctg}\frac{1}{\sqrt{R}} - \frac{R}{3} + R^2\right\} \approx 1 - \frac{4}{21R} + O\left(\frac{1}{R^2}\right), \ R \gg 1$$

2.2 Limit and critical loads and computational results

It is expedient to use a Neuber-Novozhilov type failure criterion to establish the force at which the obstacle $0 < x < a$ will rupture; it will rupture when the average normal stress over it will attain the critical value σ_{cr} – a material constant

$$\frac{1}{a}\int_{0}^{a}\sigma(x)\mathrm{d}x = \frac{\sigma_{cr}}{\Sigma_{cr}}. \tag{9}$$

where $\sigma(x) = \sigma_1(x) + \sigma_2(x)$. Then, after explicitly eliminating the singularity in $\sigma(x)$ at $x = 0$, we get the critical applied force

$$\Sigma_{cr} = \frac{\pi\sqrt{\ell}\ \sigma_{cr} - (w_0/q)\left[I_3 + I_0 I_a\right]}{I_a - 2a^{-1}\sqrt{\ell}(\sqrt{a} - \mathrm{arctg}\sqrt{a})}, \qquad I_a = \frac{1}{h}\int_{0}^{h}\frac{S(R)\mathrm{d}R}{\sqrt{R}}, \tag{10}$$

$$I_3 = \frac{3\sqrt{\ell}}{2a}\int_{0}^{r_0}\sqrt{r}i_0(r)\mathrm{d}r, \ i_0(r) = \int_{-1}^{1}\left[\frac{T(\xi)}{D+\xi}\right]^{3/2}\frac{D+\xi-r/3}{(D+\xi-r)^2}\mathrm{d}\xi; \quad h = \frac{a}{\ell}, r = \frac{x}{L}, r_0 = \frac{a}{L}$$

It should be noted that Σ_{cr} decreases when w_0 increases from nothing to w_*, because the WZ increases the stress level in the vicinity of the obstacle. The limit load Σ_* at which the WZ becomes the nucleus of a cohesive crack is governed by the criterion (3) and is

$$\Sigma_* = \frac{w_*}{qW_*} \approx \frac{3w_*(1+x_m)I_1}{q\sqrt{L}}(R_m \gg 1), I_1 = \int_{-1}^{1}\frac{\sqrt{T(\xi)(\xi+D)}}{(1+\eta^2+2\eta\xi)^2}(1+\eta\xi)\mathrm{d}\xi, x_m = L(D-\eta). \tag{11}$$

The distribution of stress $\sigma(x)$ ahead of the long crack is shown in Figures 2 and 3 for $a = 0.01,\ L = 0.01,\ 0.02,\ 0.05$, and $a = 0.1,\ L = 0.01$, respectively. From Figure 2 we notice that the peak stress at the left tip of WZ depends weakly on its half-length L; calculations not reported here show that this value is primarily determined by the width of the obstacle a. The stress distribution over the obstacle $0 < x < a$ is greatly influenced by a closely located WZ. The influence is localised only when the WZ is much farther removed from the long crack tip ($a \gg L$), as can be clearly seen on Figure 3.

From the solution (1)–(3) one can calculate the σ-w deformation relationship over the WZ domain. This relationship appears to be close to a linear function with a small distortion in the vicinity of the WZ tips.

3. Conclusions

A semi-inverse method has been proposed for solving the problems of elastic bodies containing adhesively-bonded weak zones (WZs) situated in highly asymmetric stress fields. It is based on prescribing the displacement jumps in the WZ in the form of a two-parameter basis function (i.e. a special dislocation) that is asymmetric and has the expected behaviour of the stress and displacement fields near the ends of the WZ. The advantages of this method are that it is independent of the physical origin of the adhesive forces and is simple to use for the solution of crack-WZ interaction problems. It is an approximate method which can however be improved by including higher order terms in the basis function.

The method was demonstrated on the problem of interaction between a long crack and a short WZ separated by an obstacle. Two limit states of this problem were analysed: (i) the moment at which the WZ becomes the nucleus of a cohesive crack, and (ii) the moment at which the barrier ruptures allowing coalescence of the crack with the WZ. These limit states are important for understanding the phenomenon of intermittent crack growth in structured materials. Explicit expressions have been obtained for limit loads in terms of the given or calculable parameters of the problem. Computational results did not reveal any physical inconsistency in the proposed dislocation model of the WZ, thus providing a justification for the semi-inverse method.

4. References

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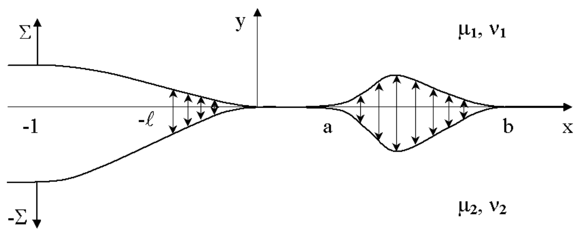

Figure 1: A semi-infinite crack subjected to wedge forces Σ at the interface between two dissimilar elastic half-planes with a small cohesive zone of length ℓ separated from an adhesively-bonded collinear weak zone (WZ) of length $2L = (b - a)$ by a microstructural obstacle of length a. All lengths ℓ, a, b, L are $\ll 1$.

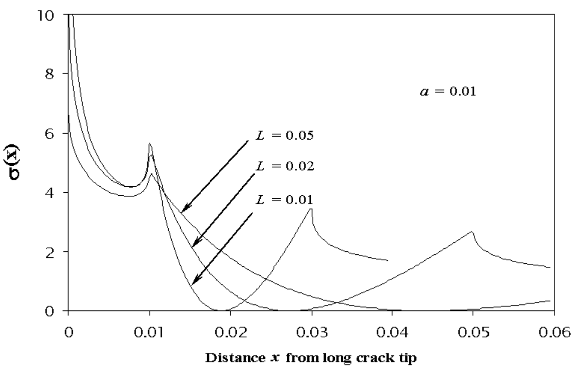

Figure 2: Variation of normal stress $\sigma(x)$ ahead of the long crack tip with the size of the weak zone when the obstacle is very narrow ($a = 0.01$). Note that the effect of WZ is felt over a large region.

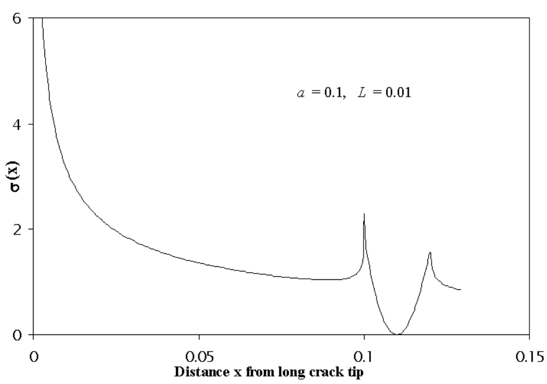

Figure 3: Variation of normal stress $\sigma(x)$ ahead of the long crack tip when the small weak zone is farther removed from it ($a = 0.1$). Note that the effect of WZ in this case is fairly localised.