SOLUTIONS FOR NONLINEAR LATTICES

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ABSTRACT

Discrete two-dimensional lattices with bistable bonds are considered. Initially, Hooke's law is valid as the first stable branch of the force–elongation diagram; then, when the elongation becomes critical, the transition to the other branch occurs. This transition is assumed to occur only in a line of the bonds; the bonds outside the transition line are assumed to be in the initial phase all the time. The transition is considered as a localized wave or as a propagating crack. A regular crack corresponds to the second branch having zero resistance, while if the bond breaks at a point of the second branch of a nonzero resistance, the formulation corresponds to the crack with a one-dimensional 'process zone'. The use of the discrete models leads to a closed mathematical formulation of the considered problems. It allows to determine the transition wave speed (or the crack speed), the total speed-dependent dissipation and the dissipation structure. For the piecewise linear trimeric diagram with a step-wise transition, explicit analytical solutions are found. Also such a solution is built for the crack with the process zone. For a general transition path, the problem can be reduced to an integral equation. The considerations are based on the corresponding intact-lattice self-equilibrated fundamental solutions expressed in terms of the previously obtained lattice-with-a-moving-crack fundamental solutions.

1 INTRODUCTION

Two-dimensional, periodic on x lattices are considered. The lattice knots in a line parallel to the x-axis, the line $n = 0$, are connected with those in the neighboring line $n = -1$ by identical, physically nonlinear *transition-line bonds*. The force–elongation diagram is presented in Fig. 1. The lattice half-planes, $n \geq 0$ and $n \leq -1$, are connected only by these bonds, that is, the local interaction between the half-planes is assumed. The connections between the knots in the half-planes can be any, but each half-plane represents a linear system. The latter condition can

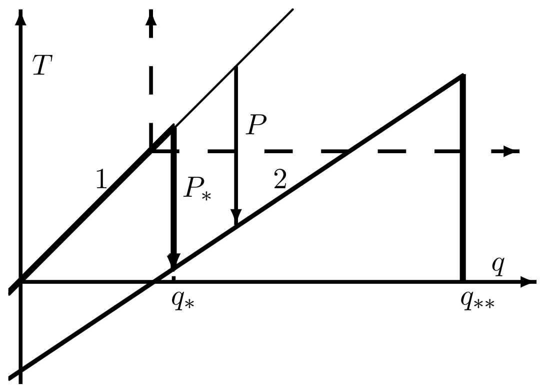

Figure 1: The piecewise linear force–elongation diagram. 1 The first, initial branch, $T = q$. 2 The second branch, $T = q - P_* - (1-\gamma)(q - q_*)$; it comes in force when the elongation reaches the critical value, $q = q_*$. The vertical distance between the branches is denoted by P. The total break of the bond occurs at $q = q_{**}$. The case of a crack corresponds to $q_{**} = q_*$, while there is a line 'process zone' if $q_{**} > q_*$. The dotted axes correspond to the diagram for an initially stressed lattice. This phase transition is assumed irreversible.

also follow from the assumption that relatively large strains may occur in the transition-line bonds only. For simplicity it is assumed that the tensile force, T, in the transition-line bond depends on the elongation, q, only. With the goal to derive an analytical solution, the dependencies, $T(q)$, shown in Fig. 1 are accepted. These dependencies are characterized by two branches, each of a constant tangent modulus. If the ratio q_{**}/q_* is large enough, such that $q < q_{**}$ all the time, and also in some other cases, a closed analytical solution can be be obtained. In the case of a diagram with a general path from the first branch to the other, the problem can be reduced to an integral equation. Infinite two-dimensional lattices and lattice strips are assumed to be under self-equilibrated loads.

Fracture based on a linear homogeneous lattice model was first analytically considered in Slepyan [1]. References to the following works and main results in this topic can be found in Slepyan [2]. Also see Marder and Gross [3]. Localized, crack-like transition wave in homogeneous bistable-bond lattices was considered in Slepyan and Ayzenberg-Stepanenko [4]. Well-known Frenkel-Kontorova model (Frenkel and Kontorova [5]; Braun and Kivshar [5]) relates to this theme as well.

In the present paper, the more general piecewise linear diagram, Fig. 1, is considered. The steady-state dynamic problem is formulated for a phase transition wave or a crack propagating with a given speed, v, along the x-axis. In this case, the transition-line bond elongation, $q = q(\eta)$, where $\eta = x - vt$. It is assumed that the transition-line bonds are in the initial phase at $\eta > 0$; the first transition occurs at

$\eta = l$, while the final break takes place at $\eta = 0$. The value of l must be determined from the condition: $q(l) = q_*$. The solution defines the transition wave speed as a function of the lattice prestress, the total speed-dependent dissipation caused by the structure-associated radiation, and the dissipation structure.

For our goal it is convenient to reformulate this nonlinear problem as follows. Consider the linear lattice all the bonds of which are in the initial phase for any q. Introduce self-equilibrated pairs of external forces, $P(\eta)$, applied to the knots connected by the transition-line bonds. The forces are directed along the corresponding bond; they must compensate the difference in the tensile forces between the real and the initial-phase dependencies. In general, these forces depend on the elongation. The intact bonds together with the external forces act on the transition-line knots, $n = 0$ and $n = -1$, in the same way as if the bonds follow the given nonlinearity. Hence, such a reformulation does not influence the lattice dynamic behavior. The elongation caused by these forces is expressed in terms of the corresponding fundamental solution, $Q(\eta)$, which reflects the structure of the intact lattice as a whole. The fundamental solution properties and their connections with the lattice structure are of the most interest. Below the main points in deriving analytical solutions are discussed. Non-dimensional variables are used.

2 CAUSALITY PRINCIPLE FOR STEADY-STATE SOLUTION

A steady-state solution is not unique if a homogeneous solution exists related to a free wave. Uniqueness can be achieved in various ways, in particular, by the use of a rule based on the causality principle. Under this principle the solution is considered as the limit of the solution to the corresponding transient problem with zero initial conditions. In terms of the Fourier transform on η with parameter k, this consideration results in the following rule. The transformed steady-state solution can be represented as a function of two variables, k and $\mathrm{i}kv$, and the latter must be treated as the limit: $0+\mathrm{i}kv \equiv \lim(s+\mathrm{i}kv)$ $(s \to +0)$, where the parameter s reflects the Laplace transform on t (the transient solution is considered as a function of two independent variables: t and η. Details can be found in Slepyan [2].

From a physical point of view, the causality principle as stated in the above *narrow sense* says that the solution should not contain waves carrying energy from infinity (nor waves exponentially growing to infinity). This is also called the Mandelshtam principle. It corresponds to the case where no energy source at infinity is assumed. In a *broad sense*, the causality principle permits all waves to occur whose sources are prescribed by the problem formulation. In particular, some remote sources may be assumed to exist at infinity and the corresponding waves carrying energy from infinity can be present in the solution. These latter waves do not obey the above-mentioned rule. Note that such waves always appear in fracture where the energy flux to the propagating crack tip is caused by remote forces. In this case, the rule serves to guide the derivation of the other part of the solution.

3 PROBLEM FORMULATION AND GOVERNING EQUATIONS

For an initially stressed lattice it is assumed that no additional external sources exist, and the phase transition wave propagates using the energy released due to the transition. In this case, the above-mentioned compensating forces gives us a nontrivial solution. In terms of the Fourier transform the elongation of the transition-line bonds can thus be represented as

$$q^F(k) = Q^F(0 + \mathrm{i}kv, k)P^F(k)\,, \tag{1}$$

where the fundamental solution, $Q(\eta)$ corresponds to impulse forces as $P = \delta(\eta)$. Further, the forces, $P^F(k)$, must be expressed in terms of the elongation. Different expressions correspond to different types of the diagram shown in Fig. 1. Three examples are shown below.

For the trimeric piecewise linear diagram with $q_{**} = \infty$

$$P^F = P_*(0 + \mathrm{i}k)^{-1} + (1 - \gamma)\left[q_- - q_*(0 + \mathrm{i}k)^{-1}\right]\,, \tag{2}$$

where the subscripts '±' are used for the right/left sided Fourier transforms ($q_+ + q_- = q^F$) and γ is the second branch nondimensional tangent modulus (the first branch modulus is equal to one). Here $q(0) = q_*$ (for a crack at $\eta < 0$: $P_* = q_*, \gamma = 0$).

In the case of a crack with a linear fracture process zone, q_{**} is finite. We consider the simplest case with $\gamma = 1$ and take $q(0) = q_{**}, q(l) = q_*$, where the latter relation serves for the determination of l value. Now

$$P^F = q_- + P_*\left(\mathrm{e}^{\mathrm{i}kl} - 1\right)/(\mathrm{i}k)\,. \tag{3}$$

Lastly, for the crack problem with softening type of the bond we have $P_* = 0, \gamma = -q_*/(q_{**} - q_*) < 0$. In this case,

$$P^F = q_- + \frac{q_{**}}{q_{**} - q_*}\left[q_l - q_*\left(\mathrm{e}^{\mathrm{i}kl} - 1\right)/(\mathrm{i}k)\right]\,, \tag{4}$$

where q_l is the Fourier transform of $q(\eta)$ over the interval $0 < \eta < l$.

In particular, the diagram adopted in eqn (2) leads to the governing equation

$$\mathcal{L}(k)q_+ + q_- = \frac{q^0[\mathcal{L}(k) - 1]}{0 + \mathrm{i}k} \tag{5}$$

with

$$\mathcal{L}(k) = \left[1 - (1 - \gamma)Q^F(k)\right]^{-1}\,, \quad q^0 = \frac{P_{**}}{1 - \gamma}\,, \tag{6}$$

while the diagram corresponding to eqn (3) results in

$$q_+ + q_-/L = Q^F P_*\left(\mathrm{e}^{\mathrm{i}kl} - 1\right)/(\mathrm{i}k) \quad (\mathcal{L} = L \text{ if } \gamma = 0)\,. \tag{7}$$

4 PROPERTIES OF THE FUNDAMENTAL SOLUTIONS

Along with $Q(\eta)$, the fundamental solution for the lattice half-plane, $U(\eta)$, is considered. The following relations are valid:

$$U^F = -\frac{Q^F}{1-Q^F}\,, \quad L = 1 - U^F \quad \mathcal{L} = \frac{L}{\gamma L + 1 - \gamma} = [1-(1-\gamma)Q^F]^{-1}\,, \tag{8}$$

Note that these equalities allows the functions Q, U and $\mathcal{L}$ to be expressed through the function L which is known for square-cell and triangular-cell discrete lattices (Slepyan [2]).

To obtain the required conclusions we now consider an auxiliary problem with the external forces, P, and correspondingly the response, q, as growing functions

$$P = \mathrm{e}^{st-\mathrm{i}k\eta}\,, \quad s>0\,, \quad \Im k = 0\,; \quad q = Q^F(s+\mathrm{i}kv,k)\mathrm{e}^{st-\mathrm{i}k\eta}\,. \tag{9}$$

Taking into account the fact that the energy flux into the lattice

$$N = \Re\left(\overline{P}\dot{q}\right) = \mathrm{e}^{2s}\left(s\Re Q^F + kv\Im Q^F\right) \tag{10}$$

must be non-negative (and using the same way of considerations respective to U) it is found that for $v>0$ and $s>0$

$$\begin{aligned} (-\infty,0) \text{ and } [1,\infty) \not\subset Q^F\,, &\quad -\pi < \arg(kQ^F) \le 0\,,\\ (0,\infty) \not\subset U^F\,, &\quad 0 < \arg(kU^F) \le \pi\,,\\ (-\infty,0] \not\subset \mathcal{L}\,, &\quad -\pi < \arg(k\mathcal{L}) \le 0\,. \end{aligned} \tag{11}$$

The index of $\mathcal{L}$ is defined as

$$\mathrm{Ind}\,\mathcal{L}(0+\mathrm{i}kv,k) = 1/(2\pi)\,[\mathrm{Arg}\,L(k=\infty) - \mathrm{Arg}\,L(k=-\infty)] \tag{12}$$

(k runs over the real k-axis). From eqn (11) it follows that

$$-1 \le \mathrm{Ind}\,\mathcal{L}(0+\mathrm{i}kv,k) \le 0\,. \tag{13}$$

However, if the dynamic formulation is adopted, that is, if the lattice mass, concentrated at the knots or distributed, is taken into account, then

$$Q^F \to 0\,, \quad \mathcal{L} \to 1 \quad (k \to \pm\infty) \quad \text{and} \quad \mathrm{Ind}\,\mathcal{L} = 0\,. \tag{14}$$

5 SOLUTION

The problem corresponding to eqn (2) was considered in detail in Slepyan and Ayzenberg-Stepanenko [4]. We here consider the case of eqn (3). Equalities in eqn (14) allows one to factorize the function L as $L(0 + \mathrm{i}kv, k) = L_+(k)L_-(k)$ by the use of the Cauchy type integral, and

$$L_+(\mathrm{i}\infty) = L_-(-\mathrm{i}\infty) = 1\,, \quad L_\pm = \mathrm{O}(1/\sqrt{k}) \quad (k \to 0) \tag{15}$$

(the latter equation is valid for subcritical speeds). There exists a solution to the homogeneous eqn (14):

$$q_{h+} = C\left[(0 - \mathrm{i}k)L_+\right]^{-1}\,, \quad q_{h-} = CL_-/(0 + \mathrm{i}k)\,, \tag{16}$$

where C is an arbitrary constant. Further, represent eqn (14) in the form as

$$L_+q_+ + q_-/L_- = \frac{P_*}{\mathrm{i}k}\left(\mathrm{e}^{\mathrm{i}kl} - 1\right)Q^F L_+ = P_*(R_+ + R_-)\,,$$
$$R_\pm = \pm\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{Q^F(\xi)L_+(\xi)}{\xi(\xi - k)}\left(1 - \mathrm{e}^{\mathrm{i}\xi l}\right)\,\mathrm{d}\xi\,, \tag{17}$$

where $\Im k > 0$ for R_+ and $\Im k < 0$ for R_-. In particular, from this it follows that

$$q_+ = q_{h+} + R_+P_*/L_+\,, \quad q_- = q_{h-} + R_-P_*L_-\,,$$
$$q(0) = C + \frac{P_*}{\pi}\int_0^\infty\left[\frac{\sin kl}{k}\Re(Q^F L_+) - \frac{1 - \cos kl}{k}\Im(Q^F L_+)\right]\mathrm{d}k\,. \tag{18}$$

The homogeneous solution in eqn (16) reflects the remote source that forced the crack to propagate, and the constant C is uniquely defined by the corresponding energy flux (see Slepyan [2]). The other unknowns, l and v, are defined by the conditions: $q(l) = q_*$, $q(0) = q_{**}$.

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