ON THE WAVE PROPAGATION IN BONES

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ABSTRACT

An elastic chiral material (a noncentrosymmetric material) is isotropic with respect to coordinate rotations but not with respect to inversions (Cosserats [1], Lakes and Benedict [2], Lakes [3,4]). Materials may exhibit chirality on the atomic scale (quartz, sugar, biological molecules) and also on a larger scale (bones, porous materials, composites containing fibers or inclusions). Materials such as piezoelectrics, represented by tensors of fifth rank, are also chiral.

Chiral effects cannot be expressed within classical elasticity since the modulus tensor, which is fourth rank, is unchanged under an inversion. But the effects of chirality in elastic materials can be described by Cosserat elasticity. In this paper, we analyze the wave propagation in a chiral Cosserat medium. We show that this medium exhibits a strange behavior. Each response is composed of two waves, traveling with different velocities, $V_1 < V_2$. The faster wave of velocity V_2 is the usual wave found in micropolar theories. The second wave travels more slowly than the first wave. The explanation of this result is that the chirality generates substantial energy sources, which remain dormant until the faster wave acts as a transfer agent supplying energy from the dormant sources to the growing of the slow wave. However, this is not surprising because Lakes *et al*. [5] experimentally observed two kinds of longitudinal waves in bovine and human bones: a slow compressional wave and a faster longitudinal wave. The propagation of waves in a chiral Cosserat medium (a noncentrosymmetric Cosserat material) is investigated with applying Laplace and Fourier transformations. An example of infinite space with a concentrated force at the origin is presented to illustrate the theory. For each displacement, stress or couple stress component, two kinds of waves are put into evidence: a slow wave and a usual faster wave. The slow wave travels more slowly than the usual wave and may be associated with the chirality.

1. INTRODUCTION

Let us consider a chiral Cosserat medium of infinite extent with Cartesian coordinates system (x, y, z). The equations of motion in the absence of the body forces and body couples are (Eringen and Suhubi [6], Eringen [7–9], Mindlin [10, 11])

$$\sigma_{kl,k} - \rho \ddot{u}_l = 0\,, \quad m_{rk,r} + \varepsilon_{klr}\sigma_{lr} - \rho j \ddot{\varphi}_k = 0\,, \tag{1}$$

where σ_{kl} are the components of the stress tensor, m_{kl} the components of the couple stress tensor, u the displacement vector, ρ the mass density, j the microinertia, φ_k the microrotation vector which in Cosserat elasticity is kinematically distinct from the macrorotation vector $r_k = \frac{1}{2}\varepsilon_{klm}u_{m,l}$, and ε_{klm} the permutation symbol. We remember that φ_k refers to the rotation of points themselves, while r_k refers to the rotation associated with movement of nearby points. The constitutive equations (Lakes and Benedict [2], Chiroiu *et al.* [12], Teodorescu *et al.* [13]) are

$$\sigma_{kl} = \lambda e_{rr}\delta_{kl} + (2\mu+\kappa)e_{kl} + \kappa\varepsilon_{klm}(r_m - \varphi_m) + C_1\varphi_{r,r}\delta_{kl} + C_2\varphi_{k,l} + C_3\varphi_{l,k},$$

$$m_{kl} = \alpha\varphi_{r,r}\delta_{kl} + \beta\varphi_{k,l} + \gamma\varphi_{l,k} + C_1 e_{rr}\delta_{kl} + (C_2+C_3)e_{kl} + (C_3 - C_2)\varepsilon_{klm}(r_m - \varphi_m), \tag{2}$$

where $e_{kl} = \frac{1}{2}(u_{k.l} + u_{l,k})$ is the macrostrain vector. λ, and μ the Lame elastic constants, κ the Cosserạt rotation modulus, α,β,γ, the Cosserạt rotation gradient moduli, and C_i, $i = 1,2,3$, the chiral elastic constants. From the requirement that the internal energy must be nonnegative (the material is stable) we obtain restrictions on the micropolar elastic constants $0 \le 3\lambda + 2\mu + \kappa$, $0 \le 2\mu + \kappa$, $0 \le \kappa$, $0 \le 3\alpha + \beta + \gamma$, $-\gamma \le \beta \le \gamma$, $0 \le \gamma$, and any positive or negative C_1, C_2, C_3 (Gauthier [14]). We add the boundary and initial conditions according to the considered problem (Delsanto *et al.* [15]).

DEFINITION Let $\mathbb{F} = \{\sigma_{kl}, m_{kl}, u_k, \varphi_k, \quad k,l = 1,2,3\}$, be a set composed of the asymmetric tensors σ_{kl}, m_{kl}, $k,l = 1,2,3$, and the vectors u_k, φ_k, $k = 1,2,3$. We call $\mathbb{F}$ an electrodynamic state on the bounded medium, if it satisfies (1) and the associated boundary and initial conditions. We demonstrate the following theorem (Teodorescu *et al.* [13]):

THEOREM *The one–by–one transformation*

$$\hat{u}_1 = K_{10}^2(u_1 + u_2 - u_3),\ \ \hat{u}_2 = K_{11}^2(u_2 + u_3 - u_1),\ \hat{u}_3 = K_{12}^2(u_3 + u_1 - u_2),$$

$$\hat{\varphi}_1 = K_{10}^2(\varphi_1 + \varphi_2 - \varphi_3),\ \ \hat{\varphi}_2 = K_{11}^2(\varphi_2 + \varphi_3 - \varphi_1),\ \hat{\varphi}_3 = K_{12}^2(\varphi_3 + \varphi_1 - \varphi_2), \tag{3}$$

with

$$K_{10}^2 = \frac{(C_2 + C_3)^2}{4(2\mu+\kappa)(\beta+\gamma)},\ K_{11}^2 = \frac{(C_2 - C_3)^2}{4(2\mu+\kappa)(\gamma-\beta)},\ K_{12}^2 = \frac{(3C_1 + C_2 + C_3)^2}{4(3\lambda+2\mu+\kappa)(3\alpha+\beta+\gamma)},$$

transforms the elastodynamic state $\mathbb{F}$ *into another elastodynamic state* $\hat{\mathbb{F}} = \{\hat{\sigma}_{kl}, \hat{m}_{kl}, \hat{u}_k, \hat{\varphi}_k, k,l = 1,2,3\}$*, composed by the symmetric tensors* $\hat{\sigma}_{kl}$, $\hat{m}_{kl}$, $k,l = 1,2,3$*, and the vectors* $\hat{u}_k$, $\hat{\varphi}_k$*, that satisfies* (1) *and the associated boundary and initial conditions. The state* $\hat{\mathbb{F}}$ *can be decomposited in the form*

$$\hat{\mathbb{F}} = \hat{\mathbb{F}}_1 + \hat{\mathbb{F}}_2, \tag{4}$$

where $\hat{\mathbb{F}}_1 = \{\hat{\sigma}_{11}, \hat{\sigma}_{13}, \hat{\sigma}_{33}, \hat{m}_{22}, \hat{u}_1, \hat{u}_3, \hat{\varphi}_2\}$ *and* $\hat{\mathbb{F}}_2 = \{\hat{\sigma}_{22}, \hat{m}_{11}, \hat{m}_{13}, \hat{m}_{33}, \hat{u}_2, \hat{\varphi}_1, \hat{\varphi}_3\}$.

Proof. The proof is immediately. We have $\hat{e}_{kl} = \frac{1}{2}(\hat{u}_{k.l} + \hat{u}_{l,k}) = \hat{e}_{lk}$, and from (2) we obtain $\hat{\sigma}_{kl} = \hat{\sigma}_{lk}$, $\hat{m}_{kl} = \hat{m}_{lk}$. Then, we introduce (2) into (1). After a proper combination of equations, the following equations in $\hat{u} = (\hat{u}_1, \hat{u}_2, \hat{u}_3)$ and $\hat{\varphi} = (\hat{\varphi}_1, \hat{\varphi}_2, \hat{\varphi}_3)$ are found

$$(\lambda+2\mu+\kappa)\nabla\nabla\hat{u}-(\mu+\kappa)K_0^2\nabla\times\nabla\times\hat{u}+\kappa(1-K_0^2)\nabla\times\hat{\varphi}=\rho\ddot{\hat{u}}\,,$$

$$(\alpha+\beta+\gamma)\nabla\nabla\hat{\varphi}-\gamma K_0^2\nabla\times\nabla\times\hat{\varphi}+\kappa(1-K_0^2)\nabla\times\hat{u}-2\kappa(1-K_0^2)\hat{\varphi}=\rho j\ddot{\hat{\varphi}}\,,\tag{5}$$

with a coupling coefficient K_0 defined as

$$K_0^2=1+\frac{(C_1+C_2+C_3)^2}{(\lambda+2\mu+\kappa)(\alpha+\beta+\gamma)}\,.$$

We see that (5) are decoupled into two sets of equations in $\hat{\mathbb{F}}_1$, and $\hat{\mathbb{F}}_2$. Next, we will concentrate only to the set of equations corresponding to $\hat{\mathbb{F}}_1$, the other set being solved in a similar way. Introducing the dimensionless quantities

$$x'=\frac{\omega}{c_1}x\,,\ z'=\frac{\omega}{c_1}z\,,\ v_i'=\frac{\omega}{c_1}\hat{u}_i\,,\ i=1,2\,,\ \phi_2'=\frac{\mu K_0^2}{\rho j\omega^{*2}}\hat{\varphi}_2\,,\ t'=\omega t\,,\ \sigma_{ij}'=\frac{1}{\mu K_0^2}\hat{\sigma}_{ij}\,,$$

$$m_{ij}'=\frac{c_1}{\gamma\omega K_0^2}\hat{m}_{ij}\,,\ \omega^{*2}=\frac{\kappa(1-K_0^2)}{\rho j},\ c_1^2=\frac{\lambda+2\mu+\kappa}{\rho},\ i,j=1,3\,,$$

where ω is the angular frequency, and by suppressing the dashes, the equations reduce to

$$v_{1,xx}+(1-a^2)v_{3,xz}+a^2v_{1,zz}-s_4^*\phi_{2,z}=\frac{1}{s_1+s_2}\ddot{v}_1\,,$$

$$v_{3,zz}+(1-a^2)v_{1,xz}+a^2v_{3,xx}+s_4^*\phi_{2,x}=\frac{1}{s_1+s_2}\ddot{v}_3\,,$$

$$\phi_{2,xx}+\phi_{2,zz}-\frac{2c_1^2\kappa(1-K_0^2)}{\omega^2\gamma K_0^2}\phi_2+\frac{c_1^2\mu}{\omega^2\gamma}(v_{1,z}-v_{3,x})=\frac{1}{s_4}\ddot{\phi}_2\,.\tag{6}$$

where

$$s_1=\frac{\lambda+\mu K_0^2}{\rho c_1^2},\ s_2=\frac{\kappa(1-K_0^2)+\mu K_0^2}{\rho c_1^2},\ s_3=\frac{\kappa j(1-K_0^2)\omega^{*2}}{\mu K_0^2c_1^2},\ s_4=\frac{\gamma K_0^2}{\rho j c_1^2}\,,$$

$$a^2=\frac{s_2}{s_1+s_2},\ s_4^*=\frac{s_3}{s_1+s_2}\,.$$

2. SOLUTIONS

Consider a bovine bone subjected to bending (fig. 1). We solve the equations (6), by considering the Laplace and Fourier transforms (Poularikas [16])

$$\{\overline{v}_i(x,z,p),\overline{\phi}_2(x,z,p)\}=\int_0^\infty\{v_i(x,z,t),\phi_2(x,z,t)\}\exp(-pt)dt\ ,i=1,3\ ,\tag{7}$$

$$\{\tilde{v}_i(\xi,z,p),\tilde{\phi}_2(\xi,z,p)\}=\int_0^\infty\{\overline{v}_i(x,z,p),\overline{\phi}_2(x,z,p)\}\exp(i\xi x)dx\ ,\ i=1,3\ .\tag{8}$$

An eigenvalue problem is obtained by taking the solutions of the form (Kumar and Choudhary [17], Munteanu and Donescu [18])

$$W(\xi,z,p)=X(\xi,p)\exp(qz),\tag{9}$$

where $W(\xi,z,p)=\{\tilde{v}_1,\tilde{v}_3,\tilde{\phi}_2\}$. The characteristic equation becomes

$$q^6-\lambda_1q^5+\lambda_2q^4-\lambda_3q^3+\lambda_4q^2-\lambda_5q+\lambda_6=0\ .\tag{10}$$

Roots of (10) are q_i , $i=1,2,...,6$, with real part positive.

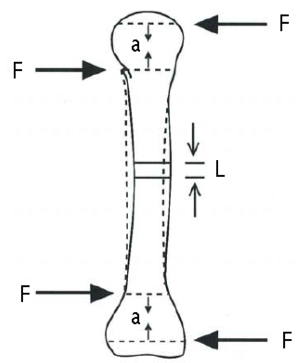

Fig. 1 A bovine bone subjected to bending.

The functions $\tilde{v}_1$, $\tilde{v}_3$ and $\tilde{\phi}_2$ are determined in the transformed domain and we can find the components of displacements, microrotations and stresses.

3. RESULTS

Using the theory of the previous sections, it can now be seen the effect of chirality in a bovine bone subjected to bending ($L=5$cm and $a=8.5$cm). The material properties are defined as

$$\rho=1800\ \text{kg/m}^3,\ \lambda=28.22\,\text{GPa},\ \mu=17.33\,\text{GPa},\ \kappa=3.9\,\text{GPa},$$

$$\alpha=7.99\,\text{GN},\ \beta=8.23\ \text{GN},\ \gamma=7.22\ \text{GN}\ ,\ j=6.02\cdot10^{-7}\ \text{m}^2,$$

$$C_1=-3.26\times10^4\,\text{N/m},\ C_2=-5.46\times10^4\,\text{N/m},\ C_3=-9.35\times10^4\,\text{N/m}\ .$$

The results show that each response is composed of two waves traveling with different velocities, $V_1 < V_2$. The faster wave of velocity V_2 is the usual wave found in micropolar theories. The second wave travels more slowly than the first wave. The explanation of this result is that the chirality generates substantial energy sources, which remain dormant until the faster wave acts as a transfer agent supplying energy from the dormant sources to the growing of the slow wave. This is not surprising because Lakes *et al*. 1983 experimentally observed two kinds of longitudinal waves in bovine and human bones: a slow compressional wave and a faster longitudinal wave.

We concentrate in this paper only to the transversal wave u_t. The transversal response is also composed from two waves, traveling with the different velocities. The wave running to the velocity V_1 is the usual wave found in micropolar theories, but the wave which travels to the velocity V_2 more slowly than the usual wave, comes only from chirality. Fig. 2 displays the transverse displacement waves at three moment of time in the region $x \in [0, L]$. This result agrees qualitatively with experimental results of Lakes *et al.* [5]. We can conclude that slow waves in bovine bones can be explained by chirality.

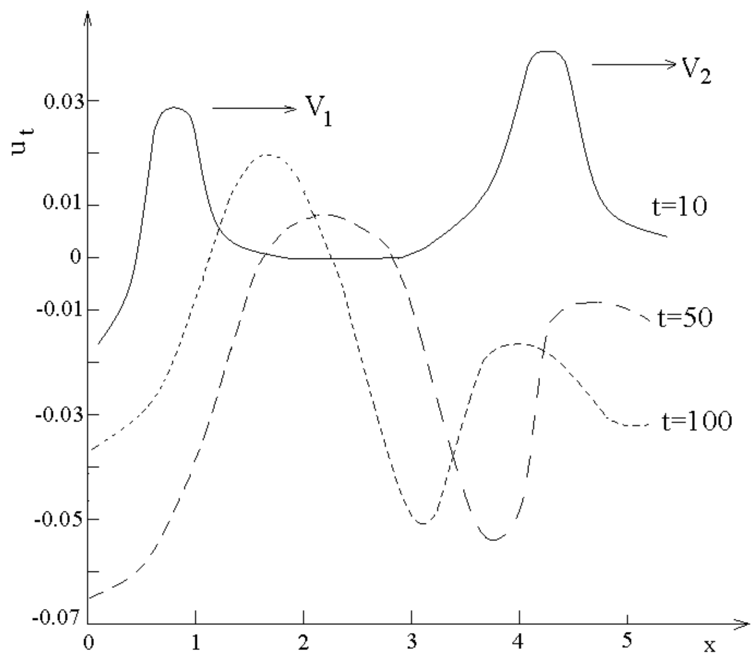

Fig. 1. Variation of two transverse displacement waves at different moments of time.

Acknowledgment. The work reported in this paper has been supported by Grant. Nr. 34395/2003, of NURC−CNCSIS of Romania

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