THE STRESS FIELD OF A SHEAR CRACK PROGAGATING IN A VISCOELASTIC MEDIUM

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ABSTRACT

The solution for a crack progagating under shear loading ("Mode II") in an isotropic viscoelastic medium with different relaxation under volume and shear deformations is summarised. The medium is infinite and the semi-infinite crack propagates along the x_1-axis at constant speed V, which may take any value up to the speed of dilatational waves. The problem is formulated for general loading but the solution is developed for loading that follows the crack, inducing a dynamic steady state. At the time of writing, the requisite Riemann–Hilbert problem has been solved and an expression for the stress component σ_{12} directly ahead of the crack has been obtained. To leading order, the stress has the same type of singularity as in the corresponding elastic problem but its strength is altered by the viscoelasticity. Higher-order asymptotic representations for the field in the vicinity of the crack (not just in its plane) are in the process of development; it is intended that explicit numerical results, suitable for comparison with experimental observation, will be included in the presentation.

1 INTRODUCTION

The motivation for this work comes from the recent laboratory demonstration (Rosakis [1]) that shear cracks can be made to propagate faster than the speed of shear waves, in polymeric materials with a weak plane. The presence of the weak plane is necessary to encourage the crack to run straight rather than deviating towards the direction of maximum tension. In the experiments, it is introduced artificially by bonding together two halves of the specimen. Such planes occur naturally in the Earth's crust and permit the shear faulting associated with earthquakes, some of which have been observed to propagate intersonically (Bouchon et al. [2]). The polymeric materials employed in the experiments display some degree of viscoelastic relaxation: measurements of moduli obtained statically differ from those obtained ultrasonically typically on the order of 10% (Rosakis [3]). There is thus some direct incentive to study the viscoelastic problem, in addition to the fact that it provides at least an example of the influence of a dissipative process on crack propagation.

The first solution for a crack propagating in a viscoelastic medium was given by Willis [4], for the case of antiplane strain. Subsequent solutions have been produced for plane strain (e.g. Walton [5] for steady-state subsonic propagation, Antipov and Willis [6] for a case of transient intersonic propagation) but only for the case that the medium has the same relaxation for volumetric or shear deformations. The present work is the first that treats the more general case, together with all speeds up to the speed of high-frequency longitudinal waves.

2 PROBLEM FORMULATION

The medium through which the crack propagates has constitutive relations

$$\sigma_{ij} = \kappa \delta_{ij} g_1 * de_{kk} + 2\mu g_2 * de'_{ij}, \tag{1}$$

where the symbol $*$ denotes convolution with respect to time and $e'_{ij} = e_{ij} - (1/3)\delta_{ij} e_{kk}$ is the deviatoric, or shear, strain. The relaxation functions $g_1(t)$, $g_2(t)$ are assumed to be convex and monotone decreasing, with $g_1(0) = g_2(0) = 1$, and tending to positive finite limits $g_1(\infty)$, $g_2(\infty)$ as $t \to \infty$. The medium is assumed to be uniform and infinite and to be loaded in some way that would generate, in the absence of the crack, a stress field $\sigma^A_{ij}(x_1, x_2, t)$. The presence of the crack which occupies, at time t, the surface

$$S(t) = \{\mathbf{x} : -\infty < x_1 < Vt, \;\; x_2 = 0, \;\; -\infty < x_3 < \infty\}, \tag{2}$$

induces *additional* stress, strain and displacement fields σ_{ij}, e_{ij}, u_i that satisfy the constitutive relations (1), the equations of motion

$$\sigma_{ij,j} = \rho \ddot{u}_i, \tag{3}$$

where ρ is the mass density, together with the boundary conditions

$$\sigma_{i2} \to -\sigma^A_{i2} \text{ as } x_2 \to \pm 0, \;\; -\infty < x_1 < Vt \tag{4}$$

and a radiation condition that this field is associated only with waves outgoing from the crack. The condition of plane strain, $u_3 \equiv 0$, is assumed.

The crack will be taken to be subject to pure Mode II loading; thus, only the "applied" traction component σ^A_{12} will differ from zero along the line of the crack (when $x_2 = 0$). Then, by symmetry, u_2 and σ_{12} will be even functions of x_2, while u_1 and σ_{22} will be odd functions of x_2. Since it has to be continuous across the plane x_2 =0 ahead of the crack, $u_1(x_1, 0, t) = 0$ ahead of the crack but it will have a discontinuity, $[u_1](x_1, t)$, across the crack surface $S(t)$. The stress component σ_{12} is continuous across the plane $x_2 = 0$ for all x_1: it must be so ahead of the crack, and is ensured by the boundary conditions (4) across the crack surface, because σ^A_{12} is continuous. The stress component σ_{22} is zero on the whole plane $x_2 = 0$ because it is an odd function of x_2, continuous across $x_2 = 0$ ahead of the crack, and from the boundary conditions (4) behind the crack, because $\sigma^A_{22} = 0$ there.

A representation for the displacement field in the half-space $x_2 > 0$ follows by relating the displacement to the traction on the boundary $x_2 = 0$, through the half-space Green's function $G_{ij}(x_1 - x'_1, x_2, x'_2, t - t')$:

$$u_i(x_1, x_2, t) = -\int\int G_{i1}(x_1 - x'_1, x_2, 0, t - t') * \sigma_{12}(x'_1, 0, t') dx'_1 dt'. \tag{5}$$

In the sequel, interest will centre on the values of u_1 and σ_{12} as the surface $x_2 = 0$ is approached from the side $x_2 > 0$. It is convenient, therefore, to streamline the notation and write $u(x_1, t)$ for $u_1(x_1, +0, t)$, $\sigma(x_1, t)$ for $\sigma_{12}(x_1, +0, t)$ and $G(x_1, t)$ for $G_{11}(x_1, 0, 0, t)$. The representation (5) then gives, when $x_2 = 0$,

$$u(x_1, t) = -(G * \sigma)(x_1, t), \tag{6}$$

the symbol $*$ now representing convolution over the relevant arguments x_1, t.

Since the crack extends with speed V, it is helpful to introduce a moving coordinate

$$x = x_1 - Vt \tag{7}$$

and to express the fields u, σ and G as functions of (x, t). Thus, for example, $u(x_1, t) = u(x + Vt, t) =: \hat{u}(x, t)$, and it is easy to check that equation (6) implies

$$\hat{u}(x, t) = -(\hat{G} * \hat{\sigma})(x, t), \tag{8}$$

the convolution being with respect to the relevant arguments x, t.

The function $\hat{u}$ is zero for $x > 0$; it is helpful to recognise this by appending a subscript so that it becomes $\hat{u}_-$. Then, similarly, decompose $\hat{\sigma}$:

$$\hat{\sigma} = \hat{\sigma}_+ + \hat{\sigma}_-, \tag{9}$$

where $\hat{\sigma}_+$ is the restriction of $\hat{\sigma}$ to the half-line $x > 0$ and is unknown, whereas $\hat{\sigma}_-$ is the corresponding restriction to $x < 0$ and is known, from the boundary condition (4).

The next step is to take the two-sided Laplace transform of equation (8), to give

$$\overline{\hat{u}}_- = -\overline{\hat{G}}[\overline{\hat{\sigma}}_+ + \overline{\hat{\sigma}}_-], \tag{10}$$

where the transform of a function $f(x, t)$ is defined so that

$$\overline{f}(\zeta, p) = \int\int \mathrm{e}^{-(\zeta x + pt)} f(x, t)\, dx\, dt.$$

Thus, the transforms $\overline{\hat{u}}_-(\zeta, p)$, $\overline{\hat{\sigma}}_-(\zeta, p)$ are analytic for $\mathrm{Re}(p) > 0$ (for causality), and in the half-plane $\mathrm{Re}(\zeta) < 0$, whereas $\overline{\hat{\sigma}}_+(\zeta, p)$ is analytic in the half-plane $\mathrm{Re}(\zeta) > 0$. In contrast, $\overline{\hat{G}}$ is defined, still for $\mathrm{Re}(p) > 0$ but only on the imaginary axis (L) in the complex ζ-plane. There is a minor complication that the domain to the left of the contour is usually called the positive side, and that to the right the negative side. With this in mind, the following definitions are made:

$$F^+(\zeta, p) = \zeta \overline{\hat{u}}_-, \quad F^-(\zeta, p) = \overline{\hat{\sigma}}_+, \quad P^+(\zeta, p) = \overline{\hat{\sigma}}_-, \quad K(\xi, p = -\zeta \overline{\hat{G}}. \tag{11}$$

The relation (8) now yields the Riemann–Hilbert problem

$$F^+(\zeta, p) = K(\zeta, p) F^-(\zeta, p) + K(\zeta, p) P^+(\zeta, p), \quad \zeta \in L, \tag{12}$$

relating the boundary values, as ζ approaches L from their respective domains of analyticity, of the functions F^+ and F^-. The reason for defining F^+ and K in the way given is to ensure that K remains bounded as $|\zeta| \to \infty$. The functions F^+ and F^- are of the same order as $|\zeta| \to \infty$ because the former is related to a strain and the latter to a stress.

The basic need, at this stage, is to obtain explicitly the transform $\overline{\hat{G}}$. This can be accomplished immediately by noting that

$$\overline{\hat{G}}(\zeta, p) = \overline{G}(\zeta, p - V\zeta), \tag{13}$$

where

$$\overline{G}(\zeta, p) = \int\int \mathrm{e}^{-(\zeta x_1 + pt)} G(x_1, t)\, dx_1\, dt. \tag{14}$$

The transform $\overline{G}(\zeta, p)$ is *exactly* like the corresponding transform of the elastic Green's function, except that the elastic wave speeds a, b of dilatational and shear waves are replaced by their viscoelastic counterparts which are given by

$$a^2(p) = p[\kappa\overline{g}_1(p) + (4/3)\mu\overline{g}_2(p)]/\rho, \ \ b^2(p) = p\mu\overline{g}_2(p)/\rho. \tag{15}$$

Substitution into (11) simply requires the replacement of p by $p - V\zeta$.

Explicitly, the function $\overline{G}$ is:

$$\overline{G}(\zeta, p) = \frac{(p^2/b^2)\beta}{\rho b^2[4\zeta^2\alpha\beta + (\beta^2 - \zeta^2)^2]}, \tag{16}$$

where

$$\alpha = (p^2/a^2 - \zeta^2)^{1/2}, \ \ \beta = (p^2/b^2 - \zeta^2)^{1/2}. \tag{17}$$

The branches of the square roots must be chosen so that $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(\beta) > 0$ for all $\zeta \in L$ ($\mathrm{Re}(p) > 0$), to ensure boundedness of the Green's function as $x_2 \to \infty$.

3 SOLUTION FOR STEADY-STATE LOADING

The limiting case of steady-state loading is obtained from the general transient case by multiplying by a factor p and then letting $p \to +0$. The relation (12) remains valid, except that the participating functions depend only on ζ; recall, however, that $K(\zeta)$ is obtained by first replacing p in $\overline{\hat{G}}$ by $p - V\zeta$ before letting $p \to +0$.

The method for solving the Riemann–Hilbert problem (12) is the usual one, which requires first solving the corresponding homogeneous problem, whose solution can be expressed in the form

$$K^+(\zeta) = K(\zeta)/K^-(\zeta), \ \ \zeta \in L. \tag{18}$$

The precise nature of the factorisation implied in (18) depends on the discontinuities of $K(\zeta)$ on L. $K(\zeta)$ is always discontinuous at infinity, and it can be discontinuous at $\zeta = 0$. However, once the factorisation is accomplished, relation (12) can be given in the form

$$F^+/K^+ - F^-K^- = P^+K^-, \ \ \zeta \in L, \tag{19}$$

which has solution

$$\begin{aligned} F^+(z) &= K^+(z)\phi(z), \ \ \mathrm{Re}(z) < 0, \\ F^-(z) &= \{K^-(z)\}^{-1}\phi(z), \ \ \mathrm{Re}(z) > 0, \end{aligned} \tag{20}$$

where

$$\phi(z) = \frac{1}{2\pi \mathrm{i}} \int_L \frac{P^+(\zeta)K^-(\zeta)}{\zeta - z} d\zeta. \tag{21}$$

The function F^-, suitably normalized, is the transform of a "weight function" that delivers the magnitude of the singularity in the stress near the crack tip. The order of the singularity follows directly from the second of equations (20), by use of the Tauberian theorem that relates the singularity at $x = 0$ to the behaviour of the transform as $|z| \to \infty$. If K^- is normalized so that

$$K^-(z) \sim \frac{1}{\Gamma(1-q)} z^{-q} \text{ as } |z| \to \infty, \tag{22}$$

then use of Plancherel's theorem, together with the Tauberian theorem, gives

$$\hat{\sigma} \sim kx^{-q} \text{ as } x \to 0, \tag{23}$$

where

$$k = -\frac{1}{2\pi \mathrm{i}} \int_L P^+(\zeta) K^-(\zeta) d\zeta = \int_{-\infty}^{0} \sigma_{12}^A(x) W(-x)\, dx, \tag{24}$$

having written $W(x)$ for the inverse transform of F^- and σ_{12}^A as a function of $x = x_1 - Vt$. The asymptotic behaviour of F^- thus carries easily-accessible physical information. A weight function for subsonic viscoelastic propagation was introduced in this way by Walton [5]; for a much more general development (explicitly for elastic material but it applies equally in the viscoelastic context), see Willis and Movchan [7], Obrezanova and Willis [8].

4 IMPLICATIONS

The structure of the Riemann–Hilbert problem (12), and the associated homogeneous problem (19), depends on what discontinuities may be displayed by $K(\zeta)$ for $\zeta \in L$. There is always a discontinuity at $\zeta = (\pm)\mathrm{i}\infty$, and there may be a discontinuity at $\zeta = 0$. Their presence, and form, depend on the speed V. It is helpful to introduce notation for the limiting wave speeds. Let

$$a_0 = [(\kappa + 4\mu/3)/\rho]^{1/2}, \;\; b_0^2 = [\mu/\rho]^{1/2} \tag{25}$$

be the "elastic" wave speeds, associated with high-frequency disturbances ($p \to \infty$), and let c_0 be the corresponding Rayleigh wave speed, associated with p/ζ giving a zero in the denominator of (16) in the limit $p \to \infty$. Let

$$a_\infty = \{[\kappa g_1(\infty) + 4\mu g_2(\infty)/3]/\rho\}^{1/2}, \;\; b_\infty = [\mu g_2(\infty)/\rho]^{1/2} \tag{26}$$

be the "low-frequency" wave speeds ($p \to 0$), with associated low-frequency Rayleigh wave speed c_∞. First, if $V < b_\infty$,

$$K(\zeta) \sim \mp \mathrm{i} A_0 \text{ as } \zeta \to \pm \mathrm{i}\infty, \tag{27}$$

where

$$A_0 = \frac{(V^2/b_0^2)(1 - V^2/b_0^2)^{1/2}}{\rho b_0^2[(2 - V^2/b_0^2)^2 - 4(1 - V^2/a_0^2)^{1/2}(1 - V^2/b_0^2)^{1/2}]}, \tag{28}$$

while

$$K(\zeta) \sim \mp \mathrm{i} A_\infty \text{ as } \zeta \to \pm 0\mathrm{i}, \tag{29}$$

where A_∞ is defined like A_0 but with the low-frequency wave speeds a_∞, b_∞ replacing a_0, b_0. If $V < c_\infty$, it is appropriate to define $K_0(\zeta)$ so that

$$K(\zeta) = \mathrm{i} A_0 (\zeta + 0)^{1/2} (\zeta - 0)^{-1/2} K_0(\zeta). \tag{30}$$

It follows that $K_0(\zeta)$ is continuous on L and tends to 1 as $\zeta \to \pm\mathrm{i}\infty$. No further detail can be given but it is now possible to factorise K_0 into a product $K_0^+ K_0^-$, and then

$$K^-(\zeta) = 2^{1/2} (\zeta - 0)^{-1/2} K_0^-(\zeta). \tag{31}$$

(The normalization is chosen here to deliver the usual Mode II stress intensity factor rather than the parameter k.) At least when $g_1 = g_2$, it is easy to prove that $K_0^-(\zeta) \equiv 1$ (Walton [5]). In that case, the weight function is independent of the viscoelastic response and the crack velocity, so the stress directly ahead of the crack is *exactly* the same as for elastostatics. This may be true generally but all that can be claimed for now is that the weight function $W(x)$ decays like $x^{-1/2}$, just as for elasticity, as $x \to \infty$.

The situation is different as soon as V exceeds c_∞, for then A_0 and A_∞ have opposite signs and $K(\zeta)$ has to be broken down differently than in (30). Full details for this, and all other speed ranges, are given in (Antipov and Willis [9]). Here, just two remarks are made. The weight function $W(x)$ has exponential decay as $x \to \infty$, when $a_\infty < V$. This can happen in the subsonic range, if $a_\infty < c_0$. The exponent q of the singularity is always the same as for the corresponding elastic problem, and depends on V when $b_0 < V < a_0$. In the intersonic range, the weight function $W(x)$ has algebraic decay as $x \to \infty$ if there is a speed range $b_0 < V < a_\infty$ but it displays exponential decay when $a_\infty < V < a_0$.

The conference presentation will include even less than this summary about the method of solution but will give more detail about the solution itself, including its form off the line of the crack, which is in the process of study at the time of writing.

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