A FAST MUTIPOLE DUAL BEM FOR THE ANALYSIS OF 2-D CRACKS

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ABSTRACT

Fast mutipole method (FMM) is applied to Dual BEM for 2-D elastic crack analysis. Dual BEM overcomes the mathematical degeneration of the displacement boundary integral equation when the two surfaces of the same crack are co-planar, by introduction of the traction boundary integral equation. The concept of finite-part integral is applied to deal with the hyper-singular integrals in traction boundary integral equation. Crack surfaces are discretized using discontinuous elements to satisfy continuity requirements. In order to achieve more run-time and space efficiency, fast mutipole method (FMM) is applied to Dual BEM. Due to the realization of mutipole expansion and local expansion, both computational complexity and memory requirement is reduced to $O(N)$, where N is the number of DOF. In mutipole expansion, a new form of complex Taylor series reduces the number of mutipole moments and local moments. Generalized minimum residual method (GMRES) is adopted as iterative solver. Sparse approximate inverse type is selected as the left preconditioner. An effective sparsity pattern is developed to reduce the cost of preconditioning and improve convergence rate. Two numerical examples are given to verify the numerical method and corresponding program. The numerical results of normalized SIF for a center crack in square plate show good agreement with that in literature. The results of COD for four cracks in square plate show good agreement with FEM results of MSC/MARC. Finally a large-scale numerical example of 3500 irregularly distributed cracks in square plate, with 1,054,824 DOF, is given. The numerical results show that the presented fast mutipole Dual BEM scheme is very efficient for large-scale crack problems.

1 INTRODUCTION

BEM is recognized as a powerful method for the study of crack problems because of its semi-analytical nature and boundary-only discretization (Cruse [1]). Compared with other numerical methods, the reduction in dimensionality dramatically reduces initial data preparation and remeshing task in crack growth. Additionally, due to the lack of internal approximations, the singular stress field around crack tip can be analyzed more accurately and more efficiently.

However, due to intrinsic difficulties, the displacement BIE degenerates when the two surfaces of one crack are considered co-planar. Dual BEM overcomes the mathematical degeneration caused by co-planar surfaces, with the displacement BIE applied for collocation on one of the two crack surfaces and the traction BIE on the other (Portela [2]). Based on Dual BEM, a single-region formulation can be built for general crack analysis.

Conventional BEM is not efficient for large-scale problems because of the dense and asymmetric matrix form. In order to reduce memory requirement and CPU time, the fast mutipole method (FMM) is applied to BEM. Barnes and Hut [3] firstly use a tree data structure and the concept of multipole expansion to calculate the matrix-vector product without forming the matrix explicitly. The computational complexity and memory requirement for matrix-vector multiplication is reduced from $O(N^2)$ to $O(N\log N)$. FMM by Greengard and Rokhlin [4] leads to the further reduction to $O(N)$ by introducing the concept of local expansion. Fast multipole BEM are investigated by many authors: Yamada et al [5] and Peirce et al [6] for 2D elastostatics, Fu et al [7] and Popov et al [8] for 3D elasticity problems, Nishimura et al [9] for crack problems of 3D

Laplace equation.

In this paper, FMM based on complex Taylor series expansion is applied to Dual BEM for 2D crack analysis. An improved sparsity pattern of sparse approximate inverse type is applied in preconditioning for GMRES solution. Large numbers of cracks in a finite plate is simulated with high efficiency and accuracy. The largest scale computed is 1,054,824 DOF.

2 BOUNDARY INTEGRAL EQUATIONS FOR CRACK ANALYSIS

Consider a 2D elastic solid with a single ideal crack. Γ_{o} is the outer boundary of the 2D solid. Γ_{c}^{-} and Γ_{c}^{+} are the two co-planar surfaces of the crack Γ_{c}. Assuming Γ_{c} is smooth and free from traction, the following displacement integral equations can be obtained.

$$c_{ab}u_b(\boldsymbol{x}) = \int_{\Gamma_{\mathrm{o}}} U_{ab}(\boldsymbol{x},\boldsymbol{y})t_b(\boldsymbol{y}) - T_{ab}(\boldsymbol{x},\boldsymbol{y})u_b(\boldsymbol{y})\mathrm{d}\Gamma(\boldsymbol{y}) - \int_{\Gamma_{\mathrm{c}}^{+}} T_{ab}(\boldsymbol{x},\boldsymbol{y})\Delta u_b(\boldsymbol{y})\mathrm{d}\Gamma(\boldsymbol{y}) \quad (\boldsymbol{x}\in\Gamma_{\mathrm{o}}) \tag{1}$$

$$\tfrac{1}{2}(u_a^{+}(\boldsymbol{x}) + u_a^{-}(\boldsymbol{x})) = \int_{\Gamma_{\mathrm{o}}} U_{ab}(\boldsymbol{x},\boldsymbol{y})t_b(\boldsymbol{y}) - T_{ab}(\boldsymbol{x},\boldsymbol{y})u_b(\boldsymbol{y})\mathrm{d}\Gamma(\boldsymbol{y}) + \int_{\Gamma_{\mathrm{c}}^{-}} T_{ab}(\boldsymbol{x},\boldsymbol{y})\Delta u_b(\boldsymbol{y})\mathrm{d}\Gamma(\boldsymbol{y}) \quad (\boldsymbol{x}\in\Gamma_c^{-}) \tag{2}$$

where $U_{ab}(\boldsymbol{x},\boldsymbol{y})$, $T_{ab}(\boldsymbol{x},\boldsymbol{y})$ stand for the displacement and traction fundamental solutions of 2D elastostatics respectively; $\boldsymbol{x}$, $\boldsymbol{y}$ are the source point and the field point respectively; u_b, t_b are displacement and traction on outer boundary Γ_{o}; Δu_a is the crack opening displacement (COD) on the crack.

Assuming continuity of both strains and tractions at point $\boldsymbol{x}$ on the crack Γ_{c}, the following traction boundary integral equation can be obtained by differentiating Eq. (2) and applying the material constitutive relationship.

$$n_a(\boldsymbol{x})\left(\int_{\Gamma_{\mathrm{c}}^{+}} S_{gab}(\boldsymbol{x},\boldsymbol{y})\Delta u_g(\boldsymbol{y})\mathrm{d}\Gamma(\boldsymbol{y}) + \int_{\Gamma_{\mathrm{o}}} S_{gab}(\boldsymbol{x},\boldsymbol{y})u_g(\boldsymbol{y}) - D_{gab}(\boldsymbol{x},\boldsymbol{y})t_g(\boldsymbol{y})\mathrm{d}\Gamma(\boldsymbol{y})\right) = 0 \quad (\boldsymbol{x}\in\Gamma_c^{+}) \tag{3}$$

After discretization on the outer boundary and the crack surface, a system of algebraic equations can be built by applying Eqs. (1-3).

Note that the kernel $S_{gab}(\boldsymbol{x},\boldsymbol{y})$ has hyper singularity of $O(r^{-2})$. The concept of finite part integral is used to deal with hypersingular integrals (Guiggiani [10]). The use of finite part integral requires that the displacement fields should be Holder continuous at source point $\boldsymbol{x}$. In this paper, the crack surfaces are discretized using discontinuous elements to satisfy the continuity requirements of the field variables.

3 FORMULATIONS OF FAST MULTIPOLE DUAL BEM (FM-DBEM)

The boundary integral of kernels can be expanded into complex Taylor series around a selected point $\boldsymbol{y}_0$. For example

$$\int_{\Gamma_j} T_{ab}(\boldsymbol{x},\boldsymbol{y})\Delta u_b d\Gamma(\boldsymbol{y}) = \sum_{k=0}^{\infty}\mathrm{Re}\left[f^{(\mathrm{r})}(\boldsymbol{x}-\boldsymbol{y}_0,k)C^{(\mathrm{fr})}(\boldsymbol{y}_0,k) + f^{(\mathrm{i})}(\boldsymbol{x}-\boldsymbol{y}_0,k)C^{(\mathrm{fi})}(\boldsymbol{y}_0,k)\right] \tag{4}$$

where x should be far enough from y to satisfy $|y-y_0| \le |x-y_0|/2$.

This operation is called mutipole expansion. $C^{(\mathrm{fr})}(\boldsymbol{y}_0,k)$ and $C^{(\mathrm{fi})}(\boldsymbol{y}_0,k)$ are mutipole moments centered at $\boldsymbol{y}_0$.

Considering the following equations

$$\int_{\Gamma} S_{gab}(\boldsymbol{x},\boldsymbol{y})u_g(\boldsymbol{y})\mathrm{d}\Gamma = C_{abmn}\frac{\int_{\Gamma} T_{mg}(\boldsymbol{x},\boldsymbol{y})u_g(\boldsymbol{y})d\Gamma}{\partial x_n(\boldsymbol{x})}, \quad \int_{\Gamma} S_{gab}(\boldsymbol{x},\boldsymbol{y})u_g(\boldsymbol{y})\mathrm{d}\Gamma = C_{abmn}\frac{\int_{\Gamma} U_{mg}(\boldsymbol{x},\boldsymbol{y})t_g(\boldsymbol{y})d\Gamma}{\partial x_n(\boldsymbol{x})} \tag{5}$$

$T_{ab}(\boldsymbol{x},\boldsymbol{y})$, $U_{ab}(\boldsymbol{x},\boldsymbol{y})$ are expanded instead of relatively complicated $S_{gab}(\boldsymbol{x},\boldsymbol{y})$, $D_{gab}(\boldsymbol{x},\boldsymbol{y})$.

In fast mutipole BEM for 2D elastostatics, there are 6 groups of multipole moments used by Yamada et al [3], but only 2 groups are used in this paper. Numerical examples show that the reduction of multipole moments can enhance the computational efficiency obviously.

If the mutipole expansion center is shifted to $\boldsymbol{y}$ from $\boldsymbol{y}_0$, the new mutipole moments can be obtained from the original ones. This translation between mutipole moments is applied when mutipole-expansion center is shifted and it is called mutipole moment to mutipole moment translation (M2M).

If the left of Eq. (4) is expanded with respect to source point $\boldsymbol{x}$ around a selected point $\boldsymbol{x}_0$, the following equation can be obtained:

$$\int_{\Gamma_j} T_{ab}(\boldsymbol{x},\boldsymbol{y})\Delta u_b \mathrm{d}\Gamma(\boldsymbol{y}) = \sum_{k=0}^{\infty} \mathrm{Re}\left[l^{(\mathrm{r})}(\boldsymbol{x}-\boldsymbol{x}_0,k) D^{(\mathrm{lr})}(\boldsymbol{x}_0,k) + l^{(\mathrm{i})}(\boldsymbol{x}-\boldsymbol{x}_0,k) D^{(\mathrm{li})}(\boldsymbol{x}_0,k) \right] \tag{6}$$

$D^{(\mathrm{lr})}(\boldsymbol{x}_0,k)$ and $D^{(\mathrm{li})}(\boldsymbol{x}_0,k)$ are local moments centered at $\boldsymbol{x}_0$. They can be obtained from the mutipole moments centered at $\boldsymbol{y}_0$. This translation from mutipole moments to local moments is called mutipole moment to local expansion translation (M2L).

If the center of local expansion is shifted to $\boldsymbol{x}_1$ from $\boldsymbol{x}_0$, the new local moments can be obtained from the old ones. This translation between local moments is applied when local-expansion center is shifted and it is called local expansion to local expansion translation (L2L).

4 NUMERICAL IMPLEMENTATION

The first step of fast mutipole algorithm is construction of a quad-tree structure. The second step is iterative process. The iterative process includes computation of mutipole moments (Upward) and computation of local moments (Downward). Upward and Downward is executed iteratively until proper accuracy is achieved. The details can be found in Greengard [4] and Yoshida [9].

In this paper, generalized minimum residual method (GMRES) is adopted as iterative solver. Sparse approximate inverse type is selected as the left preconditioner (Michele Benzi [11]). In order to reduce the cost of preconditioning and improve convergence rate, an effective sparsity pattern is developed.

For the outer boundary Γ_{o} of the 2D solid, not considering the existence of cracks, we can obtain the equation system of matrix form as follows:

$$\boldsymbol{M}_1 \boldsymbol{X} = \boldsymbol{B} \tag{7}$$

For each crack, omitting the effect of other cracks and the outer boundary, we can obtain another equation system.

$$\boldsymbol{M}_2 \Delta\boldsymbol{u} = \boldsymbol{T}^{\infty} \tag{8}$$

$\boldsymbol{M}_1^{-1}$ of Γ_{o} and $\boldsymbol{M}_2^{-1}$ of all the cracks compose the preconditioner. For cracks with identical shape and identical discretization, their $\boldsymbol{M}_2^{-1}$ must be identical in local coordinate. Thus the computational cost and memory requirement for the preconditioner is so little that can be almost ignored. Numerical example shows that the present sparsity pattern can reduce the CPU time and memory cost 40%~60%.

5 NUMERIC EXAMPLES

5.1 Example 1: test example of a center crack in a square plate

Figure 1 shows the computational model, where $w = 5\mathrm{mm}$, $s = 1\mathrm{MPa}$, the material properties are: $G = 100\mathrm{MPa}$, $n = 0.3$. The center crack is discretized to 16 discontinuous quadric elements.

The order of finite series is taken as $p = 30$. COD method is applied in computing the SIF. Table 1 shows the normalized SIF $K_1^* = K_1/(s\sqrt{p\,a})$ of the center crack for different crack sizes. Compared with the results from Isida [12], the maximum error in the BEM results is only 0.37%.

Table 1: Normalized SIF for a center crack with different crack sizes

a/w	0.1	0.2	0.3	0.4	0.5
K_1^*(Present BEM)	1.0150	1.0511	1.1248	1.2122	1.3308
K_1^*(Isida [12])	1.014	1.055	1.123	1.216	1.334

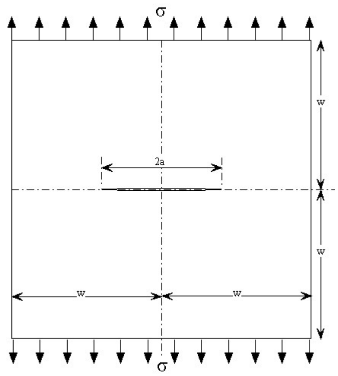

Figure 1: Center crack in a square plate

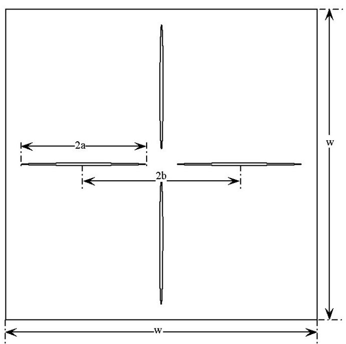

Figure 2: Four cracks in a square plate

5.2 *Example 2: test example of four cracks in a square plate*

Figure 2 shows the computational model, where $w = 10$mm, $2a = 4$mm, $2b = 5$mm, the material properties are: $G = 100$MPa, $n = 0.3$. The outer boundary of the plate is given uniform displacement in normal direction, $u_n = 1.0$. This example is analyzed by the present BEM scheme and a commercial FEM software MSC/Marc. In BEM analysis, each crack is divided into 10 discontinuous quadratic elements and the order of finite series is taken as $p = 25$. In FEM analysis using MSC/Marc, 11,890 six-node triangular elements are used totally. The COD obtained from these two numerical simulations are compared in Figure 3.

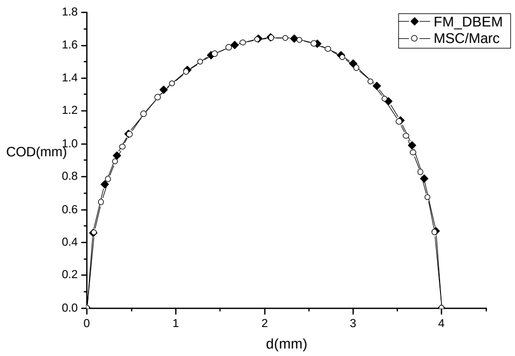

Figure 3: Comparison of COD from FM DBEM and MSC/Marc

5.3 Example 3: 3,500 irregularly distributed cracks in a square plate

In this example, 3,500 irregularly distributed cracks in a square plate as shown in Figure 4 are simulated. The number of DOF is 1,054,824. The order of finite series is 25. The total CPU time cost is 8 hours 16 minutes on one PC. Figure 5 displays the COD results of a part of the plate.

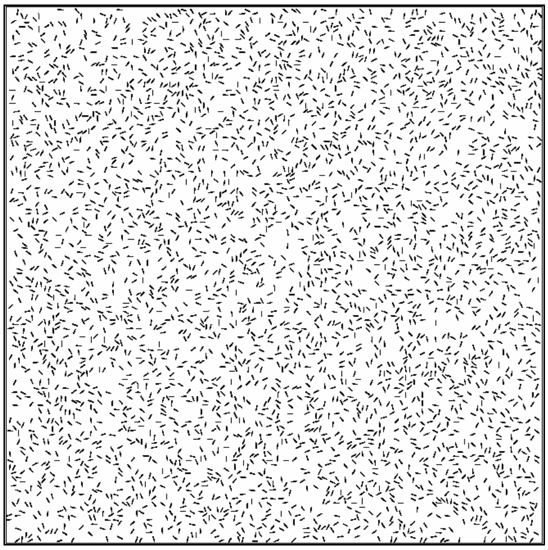

Figure 4: 3500 irregularly distributed cracks

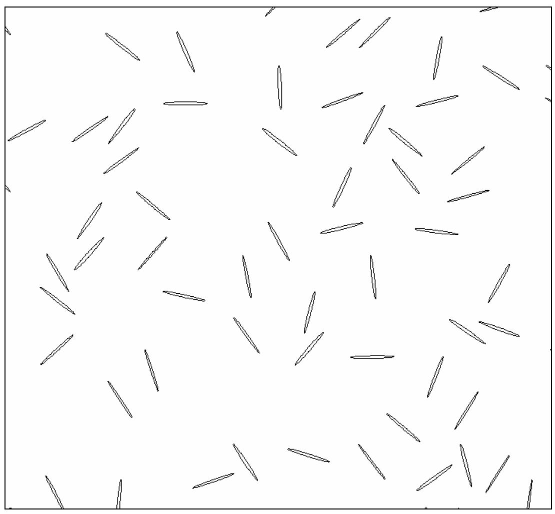

Figure 5: COD of a part of the plate

6 CONCLUSIONS

Fast mutipole Dual BEM is applied to 2D crack analysis, and both the computational complexity and memory requirement are greatly reduced. This technique is proved by numerical examples to be greatly efficient for large-scale crack problems.

A new form of complex Taylor series is used in mutipole expansion. It can not only reduce programming complexity but also enhance computational efficiency. The improved preconditioner for GMRES leads to the further reduce of CPU time and memory requirement.
In this paper, only line cracks are simulated. But the present fast mutipole Dual BEM scheme can be easily extended to cracks of various shapes. In further investigation, 3D crack problems and the simulation of crack propagation will be investigated using fast mutipole Dual BEM.

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Acknowledgements Financial support for the project from the National Natural Science Foundation of China, under grant No. 10172053 and the project from the Ministry of Education is gratefully acknowledged.