COMPUTATION OF TWO DIMENSIONAL *J*-INTEGRAL IN THE CASE OF LARGE STRAINS

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ABSTARCT

The phenomenon of failure by catastrophic crack propagation in structural materials poses problems of design and analysis in many fields of engineering. Cracks are present to some degree in all structures. They may exist as basic defects in the constituent materials or they may be induced in construction or during service life.

Using the finite element method, a lot of papers deal with the calculation of stress intensity factors for two- and three-dimensional geometries containing cracks of different shapes under various loadings to elastic bodies. In order to increase the accuracy of the results, special singular and transition elements have been used. They are described together with methods for calculating the stress intensity factors from the computed results. These include the displacement substitution method, J-integral and the virtual crack extension technique.

Despite of the large number of published finite element stress intensity factor calculations there are not many papers published on *J*-integral to elastic-plastic bodies.

At the vicinity of a crack tip the strains are not always small, but they may be large ones, too. In this case the *J*-integral can also be applied to characterise the cracks in elastic or elastic-plastic bodies.

This paper describes the computation of two dimensional *J*-integral in the case of large strains to elastic and elastic-plastic bodies and represents some numerical examples, too.

1 INTRODUCTION

Over the past decades the finite element technique has become firmly established as a useful tool for numerical solution of engineering problem and would at first sight appear to be an ideal method of studying crack behaviour in materials. In order to be able to apply the finite element method to the efficient solution of fracture problems, adaptations or further developments must be made.

Lau and his co-workers [1], [2] presented a revised *J*-estimation method under large plastic deformation. May and Kobayashi [3] investigated plane stress stable crack growth and *J*-integral using Moire interferometry to determine the two orthogonal displacements in a single edge crack specimen. Boothman and his co-workers [4] developed the *J*- and *Q*-estimation schemes for homogeneous plates. Jackiewicz [5] applied a hybrid model of steel cracking. The hybrid model uses a finite element simulation combined with an experimental test realised in the macro scale. Bouchard and his co-workers [6] demonstrated their two-dimensional local approach finite element study compared with the conventional *J*-estimation schemes and cracked body *J*-integral analysis. Saczuk and co-workers [7] presented a continuum model with inelastic material behaviour and a generalisation of the *J*-integral.

The aim of this paper is to develop a two-dimensional *J*-integral in the case of large strains for elastic and elastic-plastic material behaviour using the finite element method and to present some calculated numerical examples.

2 DEVELOPMENT OF *J*-INTEGRAL

Figure 1. shows a line integral path, which encloses the crack tip and has initial and end points, which lie, on the two crack faces. It has been shown independently by Rice [8] and Cherepanov

[9] that the following integral quantity is path independent when taken along any path, which satisfies the above conditions.

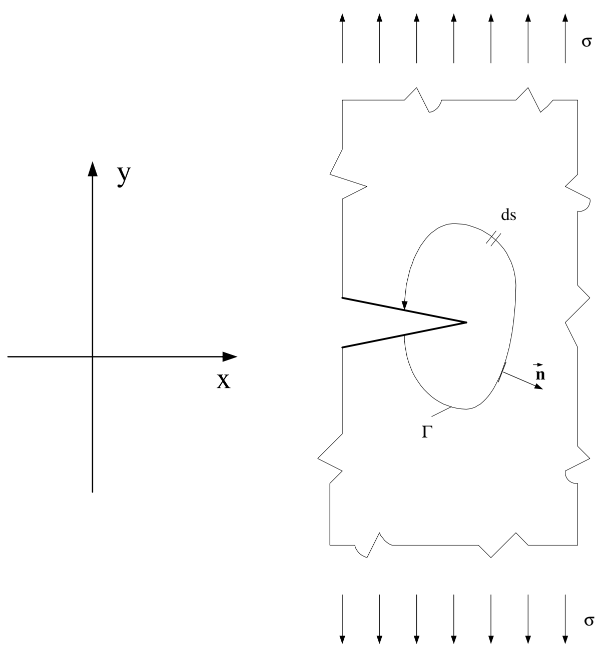

Figure1: Contour path for J_1 - integral evaluation

$$J_1 = \int_{\Gamma} \left(U n_1 - T_i \, \partial u_i / \partial x_1 \right) ds \,. \tag{1}$$

In this expression U is the strain energy density, T_i is the traction vector on a plane defined by the outward drawn normal, n_i and u_i is the displacement vector, ds is the element of arc along the path, Γ. For a closed path not containing the crack tip, $J_1 = 0$.

Knowles and Sternberg [10] noted that this expression could be considered as the first component of a vector:

$$J_k = \int_{\Gamma} \left(U n_k - T_i \, \partial u_i / \partial x_k \right) ds \,. \tag{2}$$

This integral is also path - independent provided the contour touches each surface of the crack at the tip. For elastic - plastic applications it is necessary to employ the appropriate definition of the strain energy density:

$$U = U_e + U_p . \tag{3}$$

U_e is given by

$$U_e = \frac{1}{2}\sigma_{ij}\left(\varepsilon_{ij}\right)_e, \tag{4}$$

where $\left(\varepsilon_{ij}\right)_e$ denotes the elastic components of strain. The plastic work contribution is given by

$$U_p = \int_0^{\bar{\varepsilon}_p} \bar{\sigma}\, d\bar{\varepsilon}_p\ . \tag{5}$$

In this expression $\bar{\sigma}$ and $\bar{\varepsilon}_p$ are the effective stress and effective plastic strain.

Figure 2 represents the motion of a continuum with the initial and present configurations.

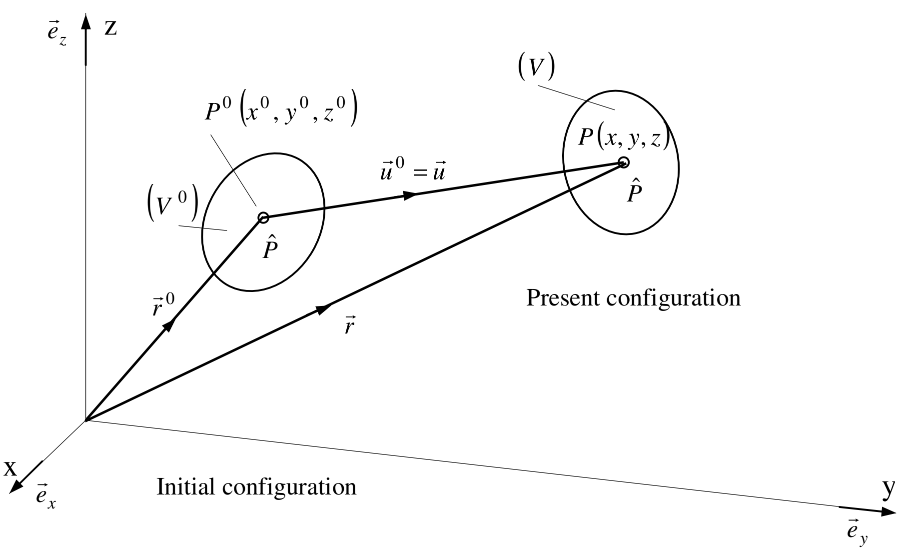

Figure 2: Motion of the continuum in the (xyz) reference coordinate-system

Let us suppose that egn (2) is valid in the present configuration in the case of large strains. As the initial configuration is known it is necessary to express the quantities in the integrand by means of $\underline{\underline{E}}^0$ Green-Lagrange strain and $\underline{\underline{T}}^0$ II.Piola-Kirchhoff stress tensors. For elastic applications it can be proved that instead of the strain energy density *U* we can write the next formula:

$$U^0 = \frac{1}{2} E^0_{kl}\, T^0_{kl}\,. \tag{6}$$

The element of arc is

$$ds = ds^0 \, \lambda_s \,, \tag{7}$$

where ds^0 is the element of arc in the initial configuration and λ_s is the stretch. The traction vector can be expressed in the next form:

$$\vec{t} = \frac{1}{\lambda_A} \underline{\underline{F}} \cdot \underline{\underline{T}}^0 \cdot \vec{n}^0 = \frac{1}{\delta \sqrt{\vec{n}^0 \cdot \left(2 \underline{\underline{E}}^0 + \underline{\underline{I}} \right)^{-1} \cdot \vec{n}^0}} \underline{\underline{F}} \cdot \underline{\underline{T}}^0 \cdot \vec{n}^0 \,, \tag{8}$$

where

$\underline{\underline{T}}^0$ - II. Piola –Kirchhoff stress tensor,

$\underline{\underline{E}}^0$ - Green-Lagrange strain tensor,

$\underline{\underline{F}}$ - strain gradient tensor,

$\underline{\underline{I}}$ - unit tensor,

$\vec{n}^0$ - the outward drawn normal in the initial configuration,

$\delta = Det \left| \underline{\underline{F}} \right|$ - Jacobian determinant.

As $\vec{u} = \vec{u}^0$, after some changes we can obtain the two components of J in two dimensions:

$$J_x = \int_{(\Gamma)} \left[U^0 \left(\frac{\partial y}{\partial x^0} d x^0 + \frac{\partial y}{\partial y^0} d y^0 \right) - \vec{t} \, \frac{\partial \vec{u}^0}{\partial x} d s^0 \, \lambda_s \right] , \tag{9a}$$

$$J_y = \int_{(\Gamma)} \left[- U^0 \left(\frac{\partial x}{\partial x^0} d x^0 + \frac{\partial x}{\partial y^0} d y^0 \right) - \vec{t} \, \frac{\partial \vec{u}^0}{\partial y} d s^0 \, \lambda_s \right] . \tag{9b}$$

It can be seen in Figure 2 that $\vec{r} = \vec{r}^0 + \vec{u}^0$, therefore we can write the next expressions:

$$d\, y = \frac{\partial y}{\partial x^0} d x^0 + \frac{\partial y}{\partial y^0} d y^0 = d y^0 + \frac{\partial u_y^0}{\partial x^0} d x^0 + \frac{\partial u_y^0}{\partial y^0} d y^0 , \tag{10a}$$

$$d\, x = \frac{\partial x}{\partial x^0} d x^0 + \frac{\partial x}{\partial y^0} d y^0 = d x^0 + \frac{\partial u_x^0}{\partial x^0} d x^0 + \frac{\partial u_x^0}{\partial y^0} d y^0 . \tag{10b}$$

The derivatives $\dfrac{\partial \vec{u}^0}{\partial x}, \dfrac{\partial \vec{u}^0}{\partial y}$ can also be expressed by means of $\vec{r} = \vec{r}^0 + \vec{u}^0$. For elastic - plastic applications it is necessary to employ the appropriate definition of the strain energy density:

$$U^0 = U_e^0 + U_{pl}^0 , \tag{11}$$

where U_e^0 is given in eqn (6) and U_{pl}^0 is similar to eqn (5):

$$U_{pl}^{0} = \int_{0}^{\overline{E}_{pl}^{0}} \overline{T}^{\,0}\, d\, \overline{E}_{pl}^{\,0} \ . \tag{12}$$

In this expression $\overline{T}^{\,0}$ and $\overline{E}_{pl}^{\,0}$ are the effective stress and effective plastic strain in the initial configuration. Using the finite element method the integration in eqn (9a) and eqn (9b) must be undertaken numerically. It can be proved that in the case of large strains the singular and transition elements can be applied, too. In the case of inclined cracks two co-ordinate systems are necessary in the initial configuration using the appropriate transformation.

3 NUMERICAL EXAMPLES

The example considered is that of a plate under tension which contains a crack of length 8 mm perpendicular to the direction of loading. The width of the plate is 20 mm and the thickness assumed to be unity. The length of the plate is 50 mm. In the first calculations the material was linear elastic with the properties E=10000 MPa, ν=0.3. The applied tensile traction was p=100 MPa. The finite element mesh represented only one quarter because of the symmetrical properties of the body (Figure 3). The finite element mesh didn't contain special elements.

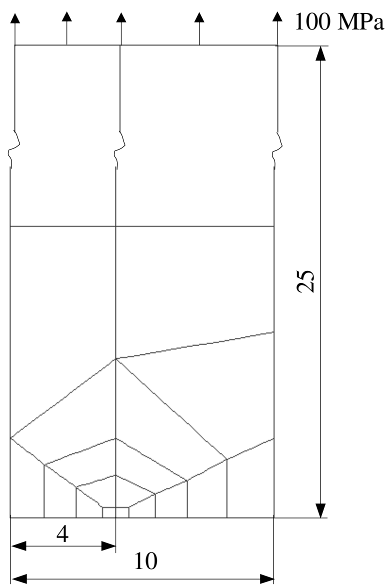

Figure 3: Finite element mesh

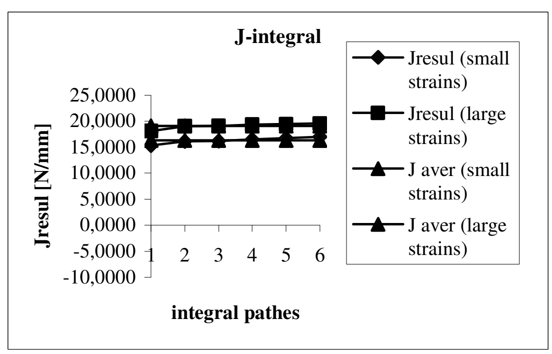

Figure 4: Calculated *J*-integral values

Theoretically J_y is zero for this problem. Figure 4 represents the calculated *J*-integral values.

In the second computations the material of the plate was linear elastic – linear hardening with H'= 0,1 E. The loading was applied in incremental steps. The increments were: 1,0; 0,1; 0,1; 0,1. The calculated J-integral values can be seen in Figure 5 for small strains and in Figure 6 for large strains.

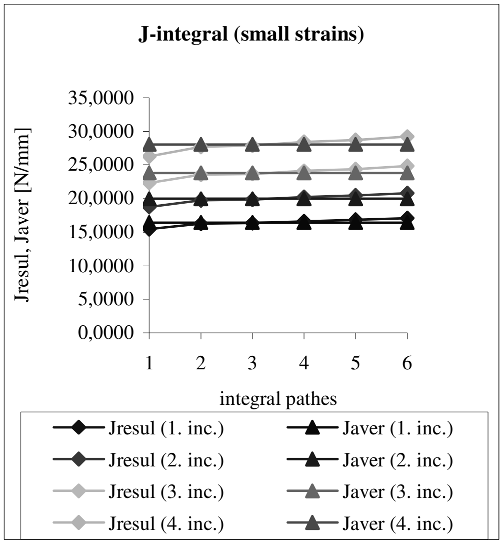

Figure 5: *J*-integral values for small strains

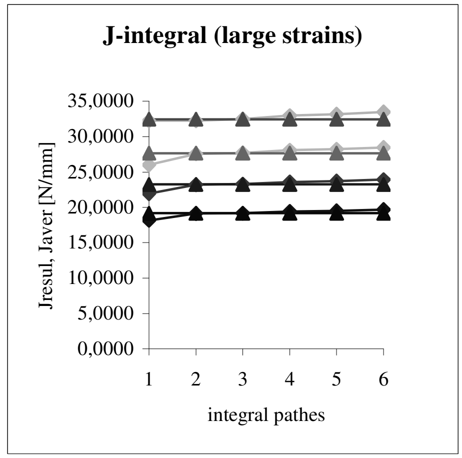

Figure 6: *J*-integral values for large strains

3 CONCLUSIONS

This paper presented the formulations and applicability of *J*-integral for large strains. The characteristics of the diagrams are very similar to those which were obtained for small strains.

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