CRACK PREDICTION IN MASONRY STRUCTURES

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ABSTRACT

The paper deals with a numerical procedure available for the analysis of unreinforced masonry structures. The model is capable of predicting the likelihood of cracking and the expected crack widths. Masonry is modelled on a discrete system of rigid blocks connected by unilateral elastic contact constraints. The numerical algorithm is based on the introduction of suitable distortion terms capable to generate internal coactions such as to give back the compatibility in the sign conditions where tensile stresses are not admissible. The distortion terms can be interpreted as cracks caused in the material because of the no-tension behaviour. A convenient way to define the ‘contact device’ which links adjacent blocks, through which a mortar joint could be simulated, is to consider a sort of Drucker’s model consisting of a set of elastic links orthogonal to the contact surface between two adjacent blocks, and an additional link tangent to the same contact surface. In accordance with the assumption that only compressive forces can be transmitted from one element to another, no tensile strength in the joints is considered. Through the results of the numerical procedure it is possible both to define and locate the cracking pattern, highlighting the actual reacting structure, within the apparent one, and to evaluate the width of the cracks located in the joints.

1 INTRODUCTION

It is well known that the main problem in the analysis of masonry structures is the different compressive and tensile strength that characterizes the material behaviour. Such a circumstance makes it impossible to understand which is, in a pre-assigned structural configuration subject to any external action (loads and/or inelastic displacements), the actual reacting structure. In order to simplify, as much as possible, structural problems involving non-isoresistant materials, the assumption of no tension behaviour for the masonry seems quite appropriate.

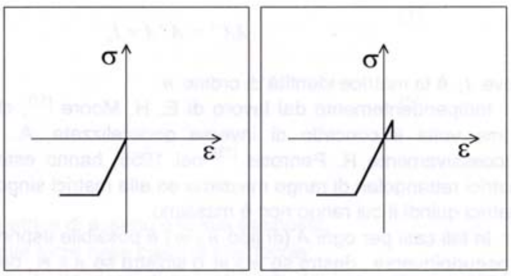


Figure 1: behaviour of contact constraint. No tension or weak tensile strength.

Such an hypothesis, that is equivalent to the statement that no tensile force can be transmitted from one portion of the structure to another, although in some case may be unrealistic, can be considered a *safe* assumption. Nevertheless, such an assumption can be considered almost exactly true if, for instance, we deal with walls, arches or domes built with stone blocks assembled dry or with very weak mortar joints. In any case it's also possible to consider a prefixed weak tensile strength (Fig.1). Generally the analysis of structural problems involving unilateral constraints, expressed through systems of equations and inequalities, requires the use of Q.P. techniques. Otherwise, as an alternative, it is possible to obtain the solution by using a step by step procedure according to which the solution relative to the *standard material* (linear elastic and bilateral) is assumed as starting point and is subsequently corrected according to the actual material skills. Such a method has already been practiced by Castigliano in 1879 [1].

2 UNILATERAL ELASTIC CONTACT CONSTRAINT

Following a typically eighteenth century idea, let us consider the general problem of a masonry structure consisting of rigid blocks linked through elastic mortar layers. In such a model the no-tension behaviour of the material is totally supposed concentrated in the mortar joint located in between two adjacent blocks. Such a joint can therefore be assumed as an unilateral elastic contact constraint. In particular, the joint can be idealized, in a Drucker's way, through an *interface device* consisting of a set of elastic links, orthogonal to the contact surface, capable of transmitting only compressive forces between the blocks, and additional links, parallel to the interface, through which the shear forces can be transmitted. The behaviour of the orthogonal links is assumed unilateral linear elastic, whereas for the parallel ones further hypotheses can be added in order to specify either the shear strength and to calibrate, for instance, the influence of the friction between the blocks, or a bilateral rigid behaviour totally capable to prevent the sliding. In practice a reasonably low number of orthogonal bars is enough to describe with significant expressiveness the behaviour of the joint and to appraise clearly the location ad depth of possible cracks (Fig. 2).

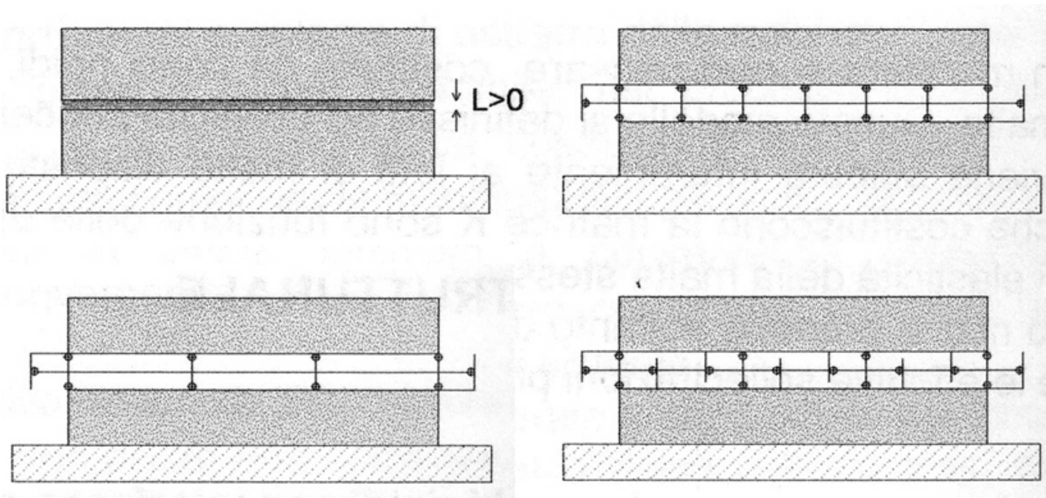


Figure2 : models of interface device. Drucker's model

3 NUMERICAL PROCEDURE

The masonry dome is modelled by a set of discrete three-dimensional rigid elements which represent single or multiple blocks of stone. Let's consider, therefore, the general problem of a masonry structure consisting of *n* rigid elements linked through *m* unilateral elastic contact interfaces. (Fig.4).

Assuming the structure subjected to the action of external loads and inelastic displacements represented respectively by the vectors $F \in \Re^{6n}$ and $\Omega_1 \in \Re^{km}$ (where the value *k* depends on the number of contact constraints chosen to characterize the interface device and defines the degree of statically indeterminacy of the structure), the problem can be expressed through a system of equilibrium and elastic-kinematical equations, with some variables, those which correspond to the unilateral links in the interface model, subjected to inequalities which express sign conditions:

$$\begin{cases} AX = F \\ A^T x + KX = \Omega_1 + \overline{\Omega}_2 \end{cases} \quad \text{sub} \quad \begin{cases} X \le 0 \\ \overline{\Omega}_2 \ge 0 \end{cases} \qquad (1)$$

In the previous form (1) $A \in \Re^{6n \times km}$ is the geometrical configuration matrix; $X \in \Re^{km}$ indicates the unknown vector of internal forces located on the interface joints; the components of the $x \in \Re^{6n}$ represents the unknown vector of displacement and rotation components of the centroids of the elements; $K \in \Re^{km \times km}$ is the diagonal stiffness matrix which characterizes the contact constraints; $\Omega_1 \in \Re^{km}$ is the vector of possible external inelastic displacements; $\overline{\Omega}_2 \in \Re^{km}$ indicates the unknown vector whose components are *internal dislocations* which need for obtaining a solution capable of satisfying both the equilibrium equations, while respecting the sign conditions, and the elastic-kinematical compatibility of the actual reacting structure. On this subject, it is convenient to distinguish, within the vector $\overline{\Omega}_2$, two types of entities, assuming for the former, related to the equilibrium aspects, the notation $\overline{\Omega}_2^*$ and for the latter, related to the compatibility ones, the notation $\overline{\Omega}_2^{**}$.

Of course the system of equations (1), subject to the first sign conditions, could also have no solution; in such a case it means that the structure cannot be equilibrated under the given system of the external actions. In this case there is no vector $X \in \Re^{km}$ which satisfies the *6n* equations and the *km* inequalities simultaneously.

However let us suppose that the system (1) in consistent. In such a case the general solution $X = X_0 + X_N$, that is able to satisfy the equilibrium problem and the first of the two inequalities, can be obtained assuming, as initial solution X_0, that is relative to the bilateral linear elastic behaviour of the contact constraints:

$$X_0 = K^{-1}A^T(AK^{-1}A^T)^{-1}F + K^{-1}(I - A^T(AK^{-1}A^T)^{-1}AK^{-1})\Omega_1 \qquad (2)$$

The initial vector X_0 can be suitably arranged in two sub-vectors: X_{0t}, whose components do not satisfy the sign conditions, and X_{0c} whose components satisfy the sign conditions:

$$X_0 = \begin{bmatrix} X_{0t} \\ X_{0c} \end{bmatrix} \tag{3}$$

Note that is $X_{0t} \in \Re^{tm}$, where *t* is the number of the contact constraints that, in the initial solution, come out stretched. In any case *t* can be greater than the degree of statically indeterminacy of the structure. Such an initial solution is then modified through the vector:

$$X_N = (I - K^{-1}A^T(AK^{-1}A^T)^{-1}A)\overline{\Omega}_2^* \tag{4}$$

which, added to X_0, satisfies the first of the (1) while respecting the sign conditions.
The properties of the orthogonal projection matrix $C = (I - K^{-1}A^T(AK^{-1}A^T)^{-1}A)$ (see [4]) and the appropriate choice of the unknown vector $\overline{\Omega}_2^*$, are the keys to understanding the meaning of the procedure. In its turn also the matrix *C* can be suitably partitioned in four sub-matrices C_t, C_1, C_1^T, C_c:

$$C = \begin{bmatrix} C_t & C_1 \\ C_1^T & C_c \end{bmatrix} \tag{5}$$

where the sub-matrix $\overline{C}_t = \begin{bmatrix} C_t & C_1 \end{bmatrix} \in \Re^{t \times km}$ has to be chosen as a full row rank matrix. On this subject the elimination of any linearly dependent row of the matrix $\overline{C}_t$, plays a key role in ascertaining the number of strictly necessary *internal dislocations* to give back the compatibility in the sign conditions. Computing the Moore-Penrose generalized inverse of $\overline{C}_t$, it is easily possible to evaluate the vector $\overline{\Omega}_2^*$:

$$\overline{\Omega}_2^* = \overline{C}_t^{-1} X_{0t} \tag{6}$$

If the solution of the unilateral problem exists, the vector solution which satisfies simultaneously the equilibrium equations and the first of the two inequalities (1), assumes the form :

$$X = \begin{bmatrix} 0 \\ X_c \end{bmatrix} \quad \text{with} \quad X_c < 0 \tag{7}$$

Since the final vector *X* is different from the first elastic vector solution X_0, it cannot satisfy, of course, the kinematical compatibility expressed through the second set of equations in the system (1).
A very easy way to build up again such a compatibility is to consider the second set of equations in the system (1) in the form $A^T\overline{x} + KX = 0$. Partitioning both the general matrix

A^T in two sub-matrices A_t^T , A_c^T , and the constitutive matrix K in K_t, K_c, we obtain the solution:

$$\bar{x} = -(A_c A_c^T)^{-1} A_c K_c X_c \quad (8)$$

which represents the vector of the displacements of the centroids of the elements only due to the actual reacting structure. Finally we can determine the vector $\overline{\Omega}_2^{**}$, so that the compatibility of the second of the (1) is already reached :

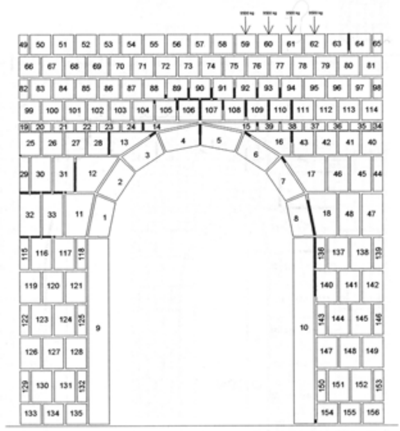
$$\overline{\Omega}_{2t}^{**} = A_t^T \bar{x} \quad (9)$$

The components of the vector $\overline{\Omega}_2^{**} \neq 0$ give the position and width of the cracks located in the joints:

$$\overline{\Omega}_2^{**} = \begin{bmatrix} \overline{\Omega}_{2t}^{**} \\ 0 \end{bmatrix} \quad (10)$$

4 NUMERICAL EXAMPLES

The efficiency and versatility of the numerical procedure has been verified through the analysis of a structure consisting of a masonry panel with an arched opening subjected to two different load condition (Fig 3, 4). For both applications, width and depth of cracks are clearly marked in correspondence of the concerned interfaces, and the deformed configuration as much clearly visualizes the structure behaviour. The procedure allows us to define the limit configuration of equilibrium and, correspondently, the collapse mechanism that is due to the formation of cracks which precedes the ruin of the structure. The correspondence between the numerical solution and some experimental results performed on tests with small brick models is totally satisfactory in terms of actual reacting structure and failure pattern.



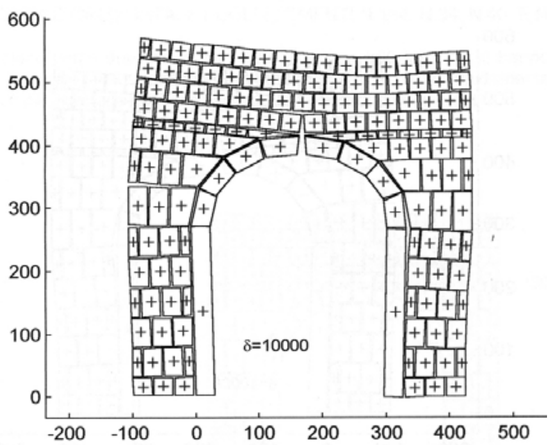


Figure 3: crack pattern and final deformation in a model of structure under eccentric loads

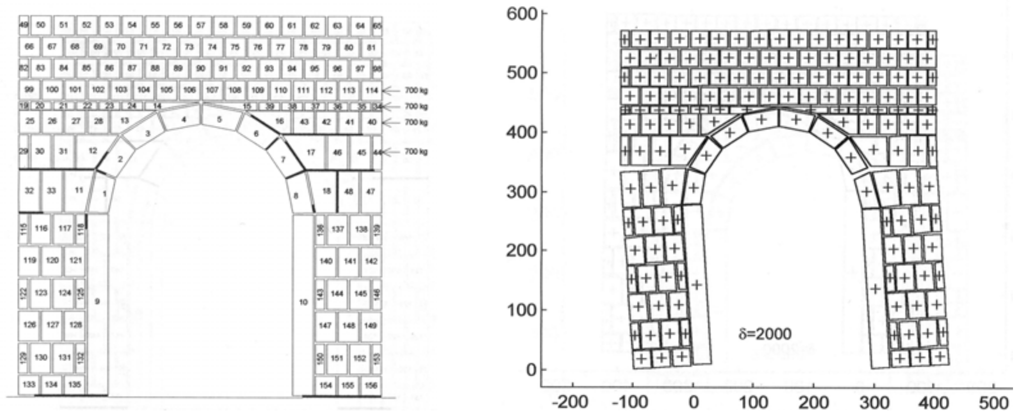


Figure 4: crack pattern and final deformation in a model of structure under lateral loads

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