APPLICATION OF PATH-INDEPENDENT INTEGRAL ANALYSIS TO FRACTURE OF MAGNETO-ELECTROELASTIC SOLIDS

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ABSTRACT

Current understanding of fracture behaviour of magneto-electroelastic solids is quite limited. A solution scheme based on the fundamental solution for a generalized edge dislocation in an infinite magneto-electroelastic solid is presented to analyze problems involving single and multiple cracks. The fundamental solution for a dislocation is obtained by extending the complex potential function formulation used in anisotropic elasticity. The solution for a continuously distributed dislocation is derived by using the solution for a single edge dislocation. The problem of a system of cracks subjected to remote loading is formulated in terms of set of singular integral equations by applying the principle of superposition and the solution for a continuously distributed dislocation. The singular integral equation system is solved by using the Chebyshev numerical integration technique. The *M*-integral for single crack and multi-cracks problems are derived and applied to the damage assessment problems in magneto-electroelastic solid. The influence of positive and negative electric and magnetic fields and crack orientation angle with respect to the poling direction are examined for both single and multiple cracks. The numerical results show that the *M*-integral presents a reliable and physically consistent measure for assessment of fracture behaviour of magneto-electroelastic materials.

1 INTRODUCTION

Magneto-electroelastic composites, such as the ferrite-ferroelectric composites, can be used to develop broadband sensors and actuating devices required for advanced engineering applications. These materials exhibit a complex and coupled magnetic-electric-mechanical field when subjected to external loading. Basic understanding of fracture behaviour of this class of materials is limited and is necessary before proceeding to study more complex issues such as magneto-electric fatigue and effects of temperature and moisture. Recently, Gao *et al.* [1], Song and Sih [2] examined the behaviour of a single crack and collinear cracks by using either the crack-tip Stress Intensity Factor (SIF) or the Strain Energy Density (SED) criterion. However, it is known that fatigue leads to the initiation of multiple arbitrarily distributed cracks rather than the appearance of a single crack or collinear cracks. In addition, the conventional crack-tip fracture parameters, such as the SIF and SED are known to have certain deficiencies when applied to assess situations involving randomly distributed clusters of cracks (Tian and Chen [3]).

Over the past forty years, many path-independent integrals have been proposed such as the *J*-integral, the J_k vectors, the *L*-integral, and the *M*-integral (Budiansky and Rice [4]). It is found that in comparison to the crack-tip based fracture criteria such as the SIF and SED, the *M*-integral presents an excellent tool when applied to assess damage in brittle materials with arbitrarily distributed and strongly interacting clusters of micro-cracks (Tian and Chen [3], Chen[5]). Unlike in the case of elastic materials, fracture problems in piezoelectric and magneto-electroelastic materials involve some fundamental issues that are not yet resolved. For example, there is no consensus on the electric boundary conditions (permeable/conducting/ insulating/presence of free charges) of a crack in a piezoelectric material and the role of electric loading on crack propagation. In addition, different fracture criteria (e.g. total energy release rate, mechanical energy release rate, strain energy density criterion, etc) often predict different behaviour and crack propagation paths.

Many of these issues applicable to piezoelectric solids are equally applicable to magneto-electroelastic solids although both experimental and theoretical studies involving fracture of magneto-electroelastic materials are limited when compared to the studies dealing with piezoelectric materials. In this paper, the authors extend the *M*-integral to magneto-electroelastic materials and apply it to damage assessment problems in a magneto-electroelastic solid containing a single crack and clusters of cracks.

2 SOLUTION FOR A CONTINUOUSLY DISTRIBUTED DISLOCATION

Following Lekhnitskii's complex potential function formulation for anisotropic elasticity (Lekhnitskii [6]), the general solutions for displacements u_1, u_2 and u_3, electric potential φ, magnetic potential ϕ, stresses $\sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{21}, \sigma_{22}$ and σ_{23}, electric displacements D_1 and D_2, and magnetic inductions B_1 and B_2 of a magneto-electroelastic medium can be expressed as (Liu e*t al.* [7]),

$$\{u_i\} = 2\,\mathrm{Re}\left[\sum_{j=1}^{5} A_{ij} f_j\,(z_j)\right] \tag{1}$$

$$\{\sigma_{2i}\} = 2\,\mathrm{Re}\left[\sum_{j=1}^{5} L_{ij} f_j^{'}(z_j)\right], \qquad \{\sigma_{1i}\} = -2\,\mathrm{Re}\left[\sum_{j=1}^{5} L_{ij}\mu_j f_j^{'}(z_j)\right] \tag{2}$$

where $\{u_i\} = \{u_1, u_2, u_3, \phi, \varphi\}^{\mathrm{T}}$, $\{\sigma_{1i}\} = \{\sigma_{11}, \sigma_{12,} \sigma_{13}, D_1, B_1\}^{\mathrm{T}}$, $\{\sigma_{2i}\} = \{\sigma_{21}, \sigma_{22,} \sigma_{23}, D_2, B_2\}^{\mathrm{T}}$; $f_1(z_1)$, $f_2(z_2)$, $f_3(z_3)$, $f_4(z_4)$, and $f_5(z_5)$ are five holomorphic functions of argument, $z_j = x_1 + \mu_j x_2$; $(\ )^{'}$ denotes differentiation with respect to z_j; $\mathbf{A}$ and $\mathbf{L}$ are two 5×5 matrices that are functions of material properties (Liu *et al.* [7]).

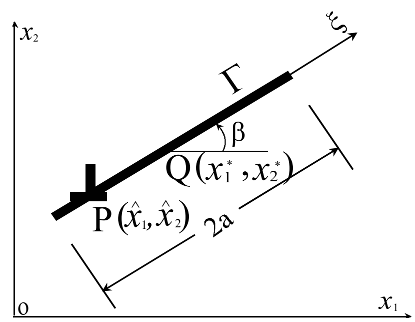

Figure 1: Coordinate system for dislocation problem.

Assume that there exists a generalized edge dislocation at point $P(\hat{x}_1, \hat{x}_2)$ in an infinite magneto-electroelastic plane, as shown in Figure 1. Following Miller [8], the complex potential functions corresponding to a single edge dislocation can be expressed as

$$f_j^{'}(z_j) = \frac{\rho_j}{z_j - \hat{z}_{0j}} \quad (j=1,2,\ldots\ldots,5) \tag{3}$$

where $\hat{z}_{0j} = \hat{x}_1 + \mu_j \hat{x}_2$ and ρ_j are complex constants.

Around a loop surrounding the point *P*, the stresses, electric displacements and magnetic induction are self-equilibrated, and the mechanical displacement, electric potential and magnetic potential jumps associated with the dislocation are denoted by the extended Burgers vector $\{\Delta u_i\} = \{\Delta u_1, \Delta u_2, \Delta u_3, \Delta\phi, \Delta\varphi\}^{\mathrm{T}}$. The complex constants ρ_j are determined by solving the following equations.

$$\mathrm{Im}\sum_{j=1}^{5} L_{ij}\rho_j = -\frac{X_i}{4\pi} \qquad \mathrm{Im}\sum_{j=1}^{5} A_{ij}\rho_j = -\frac{\Delta u_i}{4\pi} \qquad (i = 1,2,\dots,5) \tag{4}$$

where X_1, X_2 and X_3 represent the net force in the x_1, x_2 and x_3 directions around a loop surrounding the dislocation; X_4 and X_5 denote the net electric displacement and net magnetic induction on a loop surrounding the dislocation. For a standard dislocation, $X_i(i=1,2,3,4,5)$ are identical to zero.

Following Gross [9] and Xu and Rajapakse [10], the dislocation solution given by equation (3) can be used to model an arbitrarily oriented single crack in a magneto-electroelastic medium. A continuously distributed dislocation is applied along the line Γ with orientation angle β and length $2a$ (Fig. 1). A local coordinate ξ is defined along the line Γ with its origin at the center $Q(x_1^*, x_2^*)$. Let the generalized Burgers vector of a distributed dislocation along Γ be denoted by $\{\Delta u_i(\xi)\} = \{\Delta u_1(\xi), \Delta u_2(\xi), \Delta u_3(\xi), \Delta\phi(\xi), \Delta\varphi(\xi)\}^{\mathrm{T}}$. By integrating the solution for complex potential functions given by eqn (3) along Γ, the potential functions corresponding to a continuously distributed dislocation can be derived. Thereafter the coupled field in the medium can be obtained by using the equations (1) and (2).

3 ANALYSIS OF MULTIPLE CRACKS

Consider an infinite magneto-electro-elastic plane with N arbitrarily oriented cracks as shown in Figure 2. The length and orientation angle of the *kth* crack $(k = 1,2,..., N)$ are denoted by $2a_k$ and β_k respectively. The loading system is characterized by remote stresses σ_{11}^{∞}, σ_{12}^{∞} and σ_{22}^{∞}, electric fields E_2^{∞} and E_1^{∞}, and magnetic fields H_2^{∞} and H_1^{∞}. The system of cracks shown in Figure 2 can be modeled by treating each crack as a continuously distributed dislocation along the crack line. Using the solution for a distributed dislocation and the superposition principle, the multiple cracks interaction problem can be reduced to a set of singular integral equations. The system of singular integral equations can be numerically solved by expanding the dislocation densities in terms of Chebyshev polynomials and using the Chebyshev numerical integration scheme (Erdogan and Gupta [11]).

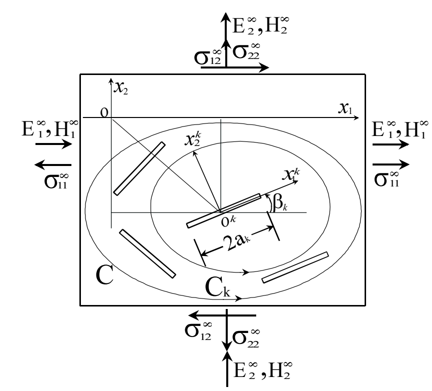

Figure 2: Geometry and loading of multi-crack system.

4 *M*-INTEGRAL ANALYSIS

The path-independent *M*-integral for a single crack in an elastic material was originally defined by Budiansky and Rice [4]. Later, Pak [12] showed that for a piezoelectric medium,

$$M = \oint_C \left[\frac{1}{2}(\sigma_{ij}\varepsilon_{ij} - D_i E_i) x_l n_l - n_i \sigma_{ip} u_{p,l} x_l - n_i D_i \phi_{,l} x_l \right] ds \tag{5}$$

In the present paper, the *M*-integral for a magneto-electro elastic medium is obtained as,

$$M = \oint_C \left[\frac{1}{2}(\sigma_{ij}\varepsilon_{ij} - D_i E_i - B_i H_i) x_l n_l - n_i \sigma_{ip} u_{p,l} x_l - n_i D_i \phi_{,l} x_l - n_i B_i \varphi_{,l} x_l \right] ds \tag{6}$$

where $i, j, p, l = 1,2$; C denotes a closed contour around the crack; n_l is the outward unit normal to C.

Consider the system shown in Figure 2. In order to evaluate the *M*-integral for a set of cracks, the closed contours C and C_k are used. The contour C encloses all cracks, while the contour C_k only encloses the *k*th crack. It can be shown that,

$$M(x_1, x_2) = \sum_{k=1}^{N} M_{C_k}(x_1, x_2) \tag{7}$$

where

$$M(x_1, x_2) = \oint_C \left[\frac{1}{2}(\sigma_{ij}\varepsilon_{ij} - D_i E_i - B_i H_i) x_l n_l - n_i \sigma_{ip} u_{p,l} x_l - n_i D_i \phi_{,l} x_l - n_i B_i \varphi_{,l} x_l \right] ds$$

$$M_{C_k}(x_1, x_2) = \oint_{C_k} \left[\frac{1}{2}(\sigma_{ij}\varepsilon_{ij} - D_i E_i - B_i H_i) x_l n_l - n_i \sigma_{ip} u_{p,l} x_l - n_i D_i \phi_{,l} x_l - n_i B_i \varphi_{,l} x_l \right] ds \tag{8}$$

5 NUMERICAL EXAMPLES AND DISCUSSIONS

In the numerical study, plane strain condition is assumed and $BaTiO_3$-$CoFe_2O_4$ is used. The remote stress field is defined by, $\sigma_{12}^{\infty} = 0, \sigma_{11}^{\infty} = 0, \sigma_{22}^{\infty} = 10 \times 10^6 N/m^2$. The material constants of $BaTiO_3$-$CoFe_2O_4$ are given by Song and Sih [2].

Single crack case

Figure 3 shows a single crack of half-length *a* and orientation angle β and the applied remote loading. Figure 4 shows the *M*-integral for three different values of β. It is clear that the *M*-integral is decreased by an applied electric or magnetic loading except in the case of a relatively low negative electric field ($-0.12\times10^5 < E_2^{\infty} < 0$ V/m) or a very low positive magnetic field. Both positive and negative electric and magnetic fields therefore generally inhibit crack propagation. A horizontal crack shows higher *M* values when compared to inclined cracks. This

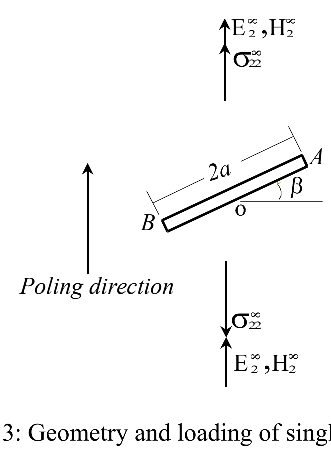

Figure 3: Geometry and loading of single crack.

implies that the crack becomes more stable as angle β increases. From a physical point of view, the limiting case of $\beta = 90°$ should be the most stable configuration under the applied loading system.

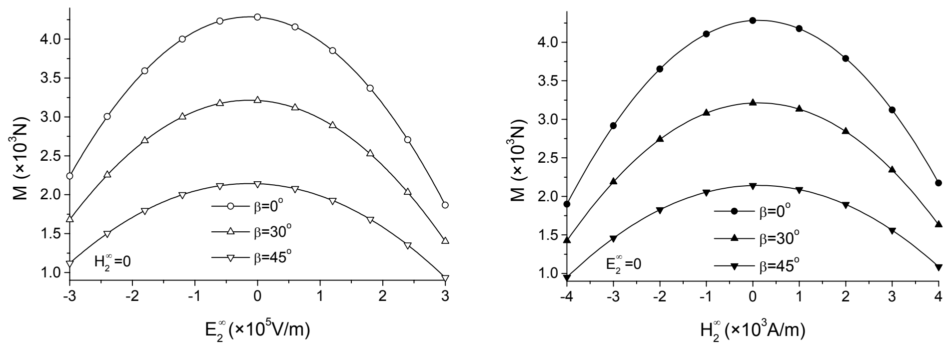

Figure 4: Variation of *M*-integral with applied electric field and magnetic field.

Multiple cracks

A case involving multiple cracks is considered to further demonstrate the merit of the *M*-integral for damage assessment. The system considered is shown in Figure 5 where an infinite plane containing four cracks of identical length 2*a* is subjected to a remote loading system. The four cracks are symmetrically placed to reduce the total number of geometric parameters governing the behaviour of the system. Numerical results are presented for the case *r*/*a*=1.5, $\theta = 45^0$ and β is changed over the range 0^o to 180^o. The *M*-integral is shown in Figure 6. It is evident that the *M*-integral shows physically more realistic results with the minimum value of *M* corresponding to $\beta = 90^o$ and the maximum values for $\beta = 0^o$ and 180^o. The dependence of *M* on electric field and magnetic is similar to that observed previously for single crack problems. On the other hand, it is found that the numerical results for the total energy and mechanical energy release rates do not capture the physical behaviour of the system.

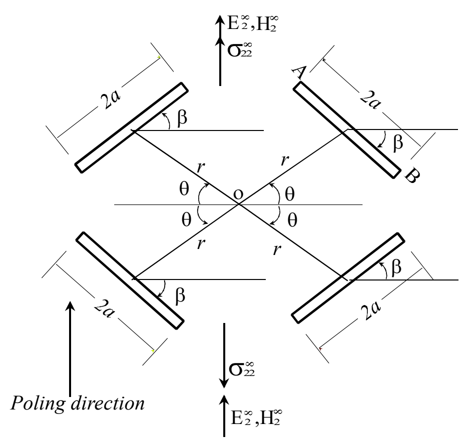

Figure 5: Geometry of system with four cracks.

6 CONCLUSIONS

It is shown that the continuously distributed dislocation method can be successfully applied to solve crack problems in magneto-electroelastic medium. The *M*-integral serves as a physically acceptable criterion for assessing damage caused by single and multiple cracks in magneto-electroelastic solid. It is found that both electric and magnetic fields generally inhibit the propagation of cracks in a magneto-electroelastic medium.

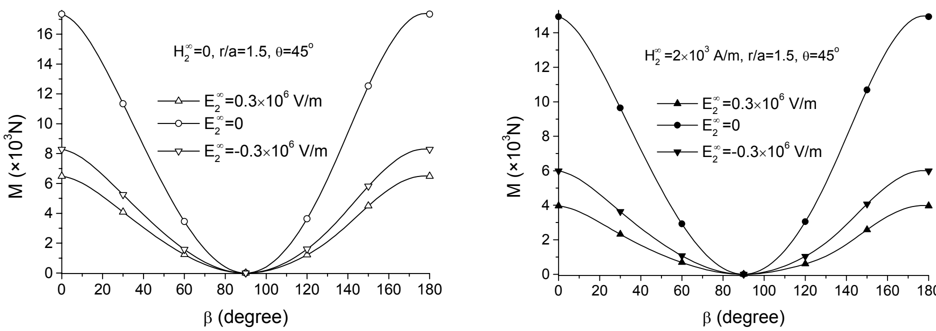

Figure 6: Variation of the *M*-integral with crack orientation angle β .

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