SCALING PROPERTIES OF THE THREE DIMENSIONAL FUSE MODEL AS A MODEL FOR BRITTLE FRACTURE

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ABSTRACT
We have implemented a numerical network of random fuses in three dimensions in order to study breakdown processes of brittle materials. With this model we have been able to measure critical exponents at breakdown, and in that sense gained information of the scaling laws involved in the morphology of fracture surfaces. We have studied both the scaling of roughness and behaviour of the correlation length ξ for broad distributions in order to examine the linkage between them.

1 INTRODUCTION

For the last twenty years there has been a growing interest for fracture morphology seen from a fundamental point of view (Mandelbrot [1]). This is quite remarkable since, as we all can see, a fracture surface is highly disordered and seems to be extremely dependent on the particular material in which the damage is sustained. Research on the general topic of fracture is nothing new, and it has interested society since ancient times. What the recent and intense studies show is that width of fracture surfaces of brittle materials scale in an anisotropic way called *self-affinity*. By this means fractal objects that scales differently in different spatial directions. By width of the fracture surface, we mean the RMS of the height $z(x, y)$, where we regard the surface in a Cartesian coordinates, written as

$$w = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(z_i - \langle z \rangle)^2}, \tag{1}$$

where we have a total of $N = L_x \times L_y$ number of surface points.

Because of the *self-affinity* of the crack surface, we can write a scaling invariance of the height in three dimensions

$$p(z; x, y) = \lambda^{\zeta} p(\lambda^{\zeta} z; \lambda x, \lambda y), \tag{2}$$

where p is probability density for a height z in position (x, y), given that $z = 0$ in $(x, y) = (0, 0)$.

An experimentally important consequence of this is that width w of a fracture surface scales

$$w \sim L^{\zeta}, \tag{3}$$

where L is the linear size of the fracture plane and ζ is denoted as the roughness exponent. When the idea of the roughness exponent was introduced, one believed that each kind of brittle material had its own and unique exponent. However, experimental studies have measured the roughness exponent to be very close to 0.8 for a large class of materials. This suggested that not only is there such a scaling exponent present, it is also universal. The hypothesis of *universal* scaling of brittle fracture was proposed about fifteen years ago (Bouchaud [2]).

2 SCALING RELATIONS

Theoretical understanding of of the mechanisms behind this universal scaling has been a real challenge to face. Recently, it was suggested by Hansen and Schmittbuhl that it has its origin in the fracture process being a correlated percolation process (Hansen [3]). This means that there exist a critical amount of damage a system can sustain at which point it breaks apart, analogous to i.e.

a melting point where thermal fluctuations measured in temperature are the driving criterion. The essence of the percolation argument is based on existence of a localization length l of the fracture, and a correlation length ξ of the entire system that grows during the breakdown process. A diverging correlation length near critical points, is the essence of qualitative phase transition, which means that we move our point of view from a microscopic one to a macroscopic. Something we observe with our own eyes. The localization length of the crack depends on the disorder in the material: Stronger disorder means larger localization length because there are no obvious weak areas of the material where the crack front can propagate more easily. For correlation lengths ξ much smaller than the localization length l, Hansen and Schmittbuhl (Hansen [3]) assumed a relation

$$\xi \sim |p - p_c|^{-\nu} , \tag{4}$$

where p is the local damage density and p_c is the damage density at failure. The phase transition in this case is density driven, meaning that a final density of damage must be sustained by the system in order to get a breakdown. This relation is taken directly from percolation theory. The reason it is only valid for large localization lengths l is that p is assumed to be spatially stationary (meaning that the statistical distribution of p-values is independent of position). The correlation length exponent ν has the value 4/3 in two-dimensional percolation and 0.88 in three-dimensional percolation (Stauffer [4]). It is by no means given that ν should be the same in the brittle fracture problem — and Toussaint and Pride suggest that it is equal to 2 (Toussaint [5]). However, it was suggested by Hansen and Schmittbuhl based on numerical simulations that the two-dimensional fuse model has $\nu = 4/3$, i.e. the value found in the pure percolation problem. When the correlation length approaches the localization length l, gradients develop in the damage — p can no longer be regarded as spatially stationary — and using arguments from gradient percolation (Sapoval [6]), Hansen and Schmittbuhl suggested the relation

$$\zeta = \frac{2\nu}{1 + 2\nu} . \tag{5}$$

3 RANDOM FUSE MODEL

As computer power has increased dramatically in the last years, there has been created a numerous of different numerical models whose aim is to verify scaling relations through computer simulations. By adjusting both size and disorder, we are able to calculate critical exponents and there relations to each other. One simple and powerful model is the random fuse network model. As the name indicates, it contains of a network of fuses with randomly distributed thresholds in two or three dimensions. The fuses act as ohmic resistors with equal resistance, but when electric current i through one fuse exceeds its threshold t it turns into an insulator irreversibly. A voltage drop is introduced over the network of fuses and the current distribution through the network is calculated through a set linear equations derived from Kirchhoff's equations at each junction between fuses.

The fuse model that we study consists of an oriented simple cubic lattice with size $L \times L \times L$. As in (Batrouni [7]), we use *periodic boundary conditions in all directions* and the average current flows in the (1,1,1)-direction. The threshold values are drawn from a spatially uncorrelated probability density $p(t)$. A voltage drop equal to unity is set up across the lattice along a given plane orthogonal to the (1,1,1)-direction. The fact that boundary conditions in all directions gives an extra dimension in space means that we must introduce an artificial voltage drop over the joint line in the current direction to run our calculations. The currents are then calculated using the Conjugate Gradient algorithm. After the currents i have been determined, the bond having the largest ratio $\max(i)/\max(t)$ is determined. This bond is then removed and the currents are recalculated until the network has fallen apart and is unable to carry any total current. It is clear that fuses near areas

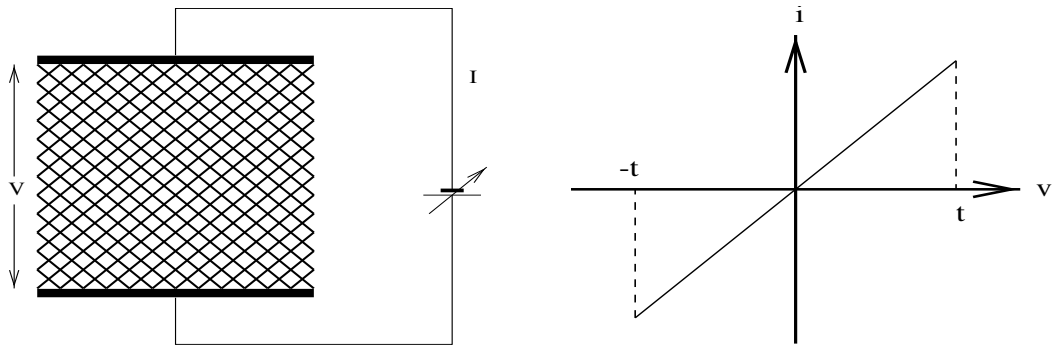

Figure 1: Schematic sketch of the fuse network with an applied current through it. On the right hand side is an illustration of the behaviour of a single fuse which turns into an insulator at high load.

of dense damage will have a heavier load and therefore be more likely to burn out during the next update of currents.

4 NUMERICAL RESULTS

It is the aim to measure ν in the three-dimensional fuse model and then derive the roughness exponent directly from eqn (5). We find the value $\nu = 0.83 \pm 0.06$, and by using eqn (5) we find $\zeta = 0.62$. Hence, the value for ν we report here is consistent with the roughness exponent measured in (Batrouni [7]). We point out that eqn (2) is not found for pure percolation in a gradient. However, the fuse model does show this behaviour making the stress-weighted percolation process in this model different from the pure percolation problem.

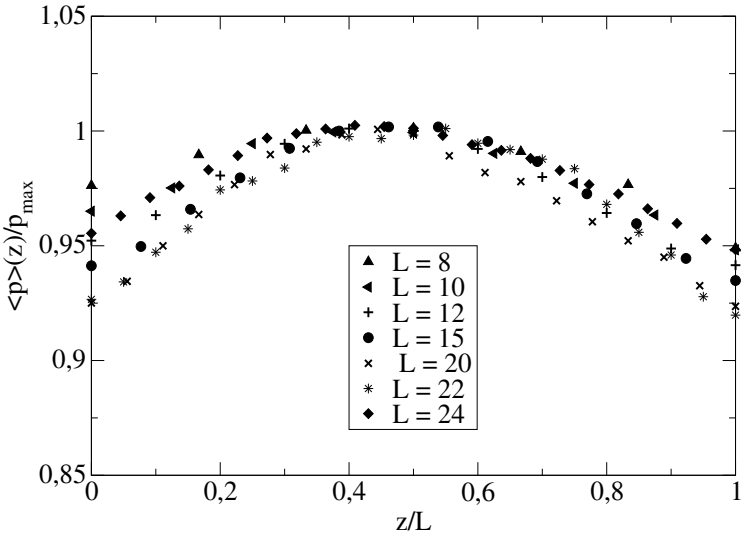

Figure 2: Normalized histogram over damage distribution averaged over a plane orthogonal to the (1,1,1)-direction for $D = 10$. The damage zone is centred and averaged according to the *center of mass*. The total damage has a weak quadratic shaped maximum.

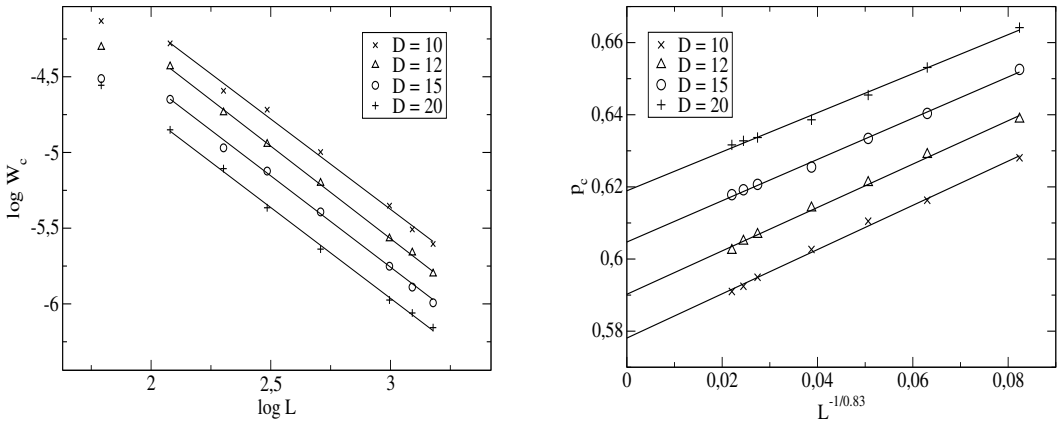

Figure 3: **a)** Plot of logarithm of the critical damage fluctuations $W_c = (\langle p^2 \rangle - \langle p \rangle^2)^{1/2}$ versus logarithm of system size L. The disorder is hence D = 10, 12, 15 and 20 respectively, and the data for each different level of disorder are shifted in order to separate them. The slopes are in that order for $D = 10$: 1.19, $D = 12$: 1.22, $D = 15$: 1.20 and $D = 20$: 1.20. Their mean is $1/\nu = 1.20 \pm 0.06$, giving $\nu = 0.83 \pm 0.06$.
b) On the right $\langle p_c \rangle$ is plotted against $L^{-1/\nu}$ with $\nu = 0.83$ with different disorder values D. As $L \to \infty$ the straight lines extrapolate to finite threshold values indicating that the system do not have a localization length smaller than the system length.

In order to measure the critical correlation length exponent ν, we must ensure a spatial invariance of the density of broken bonds. A significant concentration of broken bonds in a localized regime will cause the density of broken bonds to drop to zero as the system size increases. As a consequence we must choose a very broad distribution of thresholds for the fuses.

The threshold values t are constructed by setting $t = r^D$, where r is drawn from a uniform distribution on the unit interval. This corresponds to a probability density $p(t) \propto t^{-1+\beta}$ on the interval $0 < t < 1$ with $\beta = 1/D$. The parameter $D > 0$ controls the width of the distribution: Larger values of D corresponds to stronger disorder. In order to ensure that our results are obtained in the strong disorder phase of the fuse model, we studied $D = 10$, 12, 15 and 20. Our system sizes varied from $L = 6$ to 24 with 20000 samples generated for the smallest sizes to 2000 samples for the largest sizes.

With $D = 20$, the smallest threshold values generated are of the order $(24^3)^{-20} \approx 10^{-83}$. The system has, however, still not entered purely screened percolation regime. With this level of disorder, the system fails when a fraction of about 0.62 of the bonds have failed. The threshold values of the bonds that fail near the end of the process are about $0.62^{20} \approx 10^{-4}$ — which is of the order of the currents that are carried by the bonds in the system. Hence, there is competition between threshold values and currents, making the failure process a correlated one rather than a pure percolation one even in this seemingly extreme case.

Fig. 2 shows the damage profile in the current direction of the random fuse model with $D = 10$. Each profile is the average over thousands of samples for small L to hundreds of samples for large L. The averaging has been performed by translating each individual damage curve so that its "center-of-gravity" is placed at the same position. We denote the (1,1,1)-direction the z-direction. We define the damage as the normalized average number of burned-out fuses in the plane orthogonal to the z-direction at z. The distribution has a weak maximum in the middle. Such a maximum is much less

pronounced for the stronger disorders (i.e. larger D-values) we studied.

Following percolation analysis (Stauffer [4]), we define the survival probability Π indicating the relative number of lattices that has survived for a given average damage p. Assuming that the disorder is broad enough so that p is independent of z and there is a finite critical value of $p = p_c$ at which 50 % of the lattices survives, we have that

$$\Pi = \Phi[(p - p_c)L^{1/\nu}] \, . \tag{6}$$

This scaling ansatz implies that both the mean value of the density of broken bonds $\langle p \rangle$ and the fluctuations $(\langle p^2 \rangle - \langle p \rangle^2)^{1/2}$ at breakdown scales as $L^{-1/\nu}$ using

$$\langle p \rangle = \int p \left(\frac{d\Pi}{dp} \right) dp \, , \tag{7}$$

and

$$\langle p^2 \rangle - \langle p \rangle^2 = \int (p - \langle p \rangle)^2 \left(\frac{d\Pi}{dp} \right) dp \, . \tag{8}$$

In Fig. 3 the fluctuations of the density of broken bonds, $W_c = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$, have been plotted against the system size L. The mean value of the slopes gives $\nu = 0.83 \pm 0.06$ which gives a roughness exponent $\zeta = 0.62 \pm 0.06$ in the three-dimensional fuse model when using eqn (5). This is consistent with the previous measurements in (Batrouni [7]).

We now turn to the scaling of $\langle p \rangle$. From finite-size scaling analysis, we expect the functional dependency

$$\langle p \rangle = p_c + \frac{A}{L^{1/\nu}} \tag{9}$$

on L. We show this relation for different values of D in Fig. 3b.

This way of measuring the critical exponent ν is much less sensitive than the one presented in Fig. 3a.

5 CONCLUSION

In summary, we have determined the correlation length exponent in the three-dimensional fuse model to be $\nu = 0.83 \pm 0.06$. Furthermore, using eqn (5), this is consistent with the previously measured roughness exponent $\zeta = 0.62 \pm 0.05$ (Batrouni [7]), lending support to the scenario proposed by Hansen and Schmittbuhl (Hansen [3]) for understanding the universality of the roughness exponent in the fuse model and brittle fracture. Our analysis was based on studying the fuse model with strong enough disorder for the breakdown process to develop in a percolation-like manner with p spatially stationary so that the tools developed for studying that problem could be used in the present one. We note that in this regime, one will not see the fracture roughness scaling of eqn (2): The fracture will have a fractal structure. When, on the other hand, the disorder is weak enough for localization to set in, p is no longer spatially stationary, and the crack becomes *self-affine*.

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