ANALYSIS OF CRACK-TIP FIELDS IN FUNCTIONALLY GRADED MATERIALS

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ABSTRACT

Functionally Graded materials (FGMs) have been developed as super-resistant materials for propulsion systems and airframe of space shuttles in order to decrease thermal stresses and to increase the effect of protection from heat. It has been experimentally observed that crack in FGMs is the most common failure mode of a metal-ceramic FGM when it is subjected to some dangerous loads such as a thermal shock or mechanical shock. Therefore, it is very important to consider the thermally and mechanical induced fracture behaviors of FGMs. In this paper, a new multi-layered model for fracture analysis of functionally graded materials with arbitrarily varying elastic moduli under plane deformation has been developed. In this model, the FGM is divided into several sub-layers and in each sub-layer the reciprocal of the shear modulus is assumed to be a linear function of the depth while the poisson's ratio is assumed to be a constant. With this new model, a FGM strip containing a Griffith crack under in-plane mechanical loads is investigated. Employment of transfer matrix method and Fourier integral transform technique reduces the problem to a system of Cauchy singular integral equations which are solved numerically. Stress intensity factors of a Griffith crack of a FGM strip are then obtained.

1 INTRODUCTION

With the increasing use of functionally graded materials (FGMs) in modern technology, fracture analysis of them has been of growing interest to both scholars and engineers. In this process, the non-homogeneities of such materials should be taken into account. However, due to the mathematical difficulties arising from the fact that the material properties of FGMs vary in space, most of the current researches on the fracture analysis have been confined to some special cases. Among them Erdogan and co-workers were the first to use the model of exponential function to

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simulate the shear modulus and have investigated a series of crack problems of non-homogeneous materials (Delale and Erdogan [1, 2], Erdogan and Ozturks [3]). This model then was widely adopted to study the fracture problems of FGMs as coating or interfacial layers (Jin and Batra [4]; Chen and Erdogan [5]; Kadioğlu et al. [6]; Choi et al. [7]). Ergüven and Gross [8] employed the perturbation approach to study the crack problem of non-homogeneous materials with properties of slight variation. Recently Wang et al. [9] used a piecewise multi-layered model to study the fracture behavior of FGMs with arbitrarily varying properties. In this model, constant shear modulus in each sub-layer is assumed. This implies that the material properties involve discontinuities at the sub-interface. To overcome this disadvantages, Huang and Wang [10,11] recently suggested a new multi-layered model for the static and dynamic fracture analysis of FGMs with properties varying arbitrarily under the anti-plane and plane deformation. Based on the fact that an arbitrarily curve can be approached by a series continuous but piecewise linear curves, the FGMs are modeled as a multi-layered medium with elastic moduli varying linearly in each sub-layer and continuous on the sub-interfaces. But many difficult mathematic problems are met with this model to study the fracture problems of FGMs. To overcome the mathematic difficulties, we suggest another linear model in which the FGMs are modeled as a multi-layered medium with the reciprocal of elastic moduli varying linearly in each sub-layer and continuous on the sub-interfaces. Using this model a FGMs strip containing a Griffith crack was studied.

2 BASIC EQUATION OF THE PROBLEM

2.1 The new linear multi-layered model for fracture analysis of FGMs

Consider a FGM strip of thickness h_0. A through crack of length $2c$ lies parallel to the boundary of the strip and the strip is loaded at the crack surface, see Figure 1. Generally the shear modulus $\mu(y)$ may be described by an arbitrary continuous function of y with boundary values $\mu(h_0) = \mu_0$, $\mu(0) = \mu^*$. Wu and Erdogan [12] have shown that the influence of the variation in Poison's ratio on stress intensity factors is rather insignificant. Therefore, as they did, we assume the Poison's ratio ν is constant in the strip. Considering the fact that an arbitrary curve can be approximated by a series of continuous but piecewise linear curves, we develop a new multi-layered model as shown in Fig. 1. In this model, the functionally graded strip is divided into N sub-layers with the crack on the k th sub-interface (k may be any integer between 1 and N). The reciprocal of shear modulus of the strip varies linearly in each sub-layer and is continuous at the sub-interfaces, i.e.,

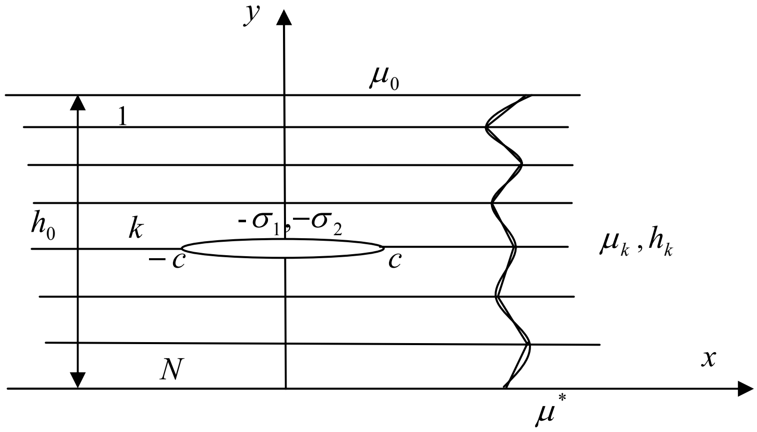

Figure 1: The new linear multi-layered model for the FGMs strip

$$\frac{1}{\mu(y)}=\frac{1}{\mu_j(y)}=\frac{1}{\overline{\mu}_j}(a_j+b_j y) \qquad h_j \le y \le h_{j-1} \tag{1}$$

where $\overline{\mu}_j$ is equal to the real value of the shear modulus at the sub-interface, $y=h_j$, i.e.,

$\overline{\mu}_j=\mu_j(h_j)=\mu(h_j)$ which leads to

$$a_j=\frac{h_{j-1}-\dfrac{\overline{\mu}_j}{\overline{\mu}_{j-1}}h_j}{h_{j-1}-h_j} \qquad b_j=\frac{\dfrac{\overline{\mu}_j}{\overline{\mu}_{j-1}}-1}{h_{j-1}-h_j} \tag{2}$$

2.2 Transfer matrix and dual integral equation

Consider the state of the plane strain deformation. The original problem may be viewed as the superposition of the following two sub-problems: (I) the functionally graded strip free of cracks is subjected to loads on its boundary, inducing shear and tensile tractions $\sigma_1(x)$ and $\sigma_2(x)$ on the crack plane; (II) the crack surface is loaded under $-\sigma_1(x)$ and $-\sigma_2(x)$ with the boundary free. Since problem (I) contributes nothing to the singular stress fields at the crack tips, we will only pay attention to problem (II) treating $-\sigma_1(x)$ and $-\sigma_2(x)$ as known functions. The governing equation for the strip can be expressed in terms of Airy stress function as

$$\nabla^4 F_j+\frac{2\mu_j'(y)}{\mu_j(y)}\frac{\partial}{\partial y}(\nabla^2 F_j)=0 \tag{3}$$

Eqn (3) can be solved by using Fourier integral transform and finally, we obtain the transform Airy stress function in each sub-layer as

$$\widetilde{F}_j = A_1\phi_{j1}(t) + A_2\phi_{j2}(t) + A_3\phi_{j3}(t) + A_4\phi_{j4}(t) \tag{4}$$

where $t = s\mu_j(y)/\mu'_j(y)$, $\phi_{j1} = e^t$, $\phi_{j2} = e^{-t}$

$$\phi_{j3} = e^t \int \frac{dt}{2ts\mu'_j} - e^{-t}\int \frac{e^{2t}dt}{2ts\mu'_j}, \quad \phi_{j4} = e^t \int \frac{e^{-2t}dt}{2ts\mu'_j} - e^{-t}\int \frac{dt}{2ts\mu'_j},$$

The displacement and stress component can be consequently given as

$$\{S_j\} = [T_j(y)]\{A_j\} = [T_{j1}(y), T_{j2}(y), T_{j3}(y), T_{j4}(y)]\{A_j\} \tag{5}$$

where $\{S_j\} = [\widetilde{u}_{xj}, \widetilde{u}_{yj}, \widetilde{\sigma}_{xyj}, \widetilde{\sigma}_{yyj}]^T$, $\{A_j\} = [A_{j1}, A_{j2}, A_{j3}, A_{j4}]^T$ and

$$[T_j(y)] = [T_{jl1}(y), T_{jl2}(y), T_{jl3}(y), T_{jl4}(y)]^T$$

with

$$T_{jl1}(y) = -\frac{i\mu_j}{s}\frac{d^2\phi_{jl}}{dy^2} - i\nu s\mu_j\phi_{jl}, \qquad T_{jl3}(y) = -is\frac{d\phi_{jl}}{dy}, \quad l = 1,2,3,4$$

$$T_{jl2}(y) = \frac{\mu_j}{s^2}\frac{d^3\phi_{jl}}{dy^3} + \frac{\mu'_j}{s^2}\frac{d^2\phi_{jl}}{dy^2} - \mu_j(2+\nu)\frac{d\phi_{jl}}{dy} + \nu\mu'_j\phi_{jl}, \quad T_{jl4} = -s^2\phi_{jl}.$$

Making use of the continuity of stresses and displacements at the sub-interface except at the crack surface where the displacement involve jumps, one may have

$$\{S_j\} - \{S_{j+1}\} = \{\Delta S_k\}\delta_{kj} \qquad y = h_j, j = 1,2,\cdots N \tag{6}$$

in which δ_{kj} is the Kronecker delta and $\Delta S_k = [\Delta\widetilde{u}_{xk}, \Delta\widetilde{u}_{yk}, 0, 0]^T$ with Δu_{xk} and Δu_{yk} being the Fourier transforms of the jumps of the displacements across the crack faces.

Eqn (6) in essence is a recurrence relation that in combination of Eqn (5) can yield $\{A_j\}$ in terms of $\{\Delta S_k\}$

$$\{A_j\} = ([\overline{L}_{jk}] + [\overline{K}_{jk}](1 - H(j-k))\{\Delta S_n\} \tag{7}$$

where $H(\cdot)$ is the heaviside function, and

$$[w_0]=I,\ [w_j]=[T_j(h_j)]^{-1}[T_j(h_j)],\ [\overline{w}_j]=[w_1]\cdots[w_j],\ [L_k]=[\overline{w}_{k-1}][T_k(h_k)]^{-1},$$

$$[L_{jk}]=[\overline{w}_{j-1}]^{-1}[\overline{w}_n][D]^{-1}[E],\ [K_{jk}]=[\overline{w}_{j-1}]^{-1}[L_k],\ [D]=\begin{bmatrix}D_1\\D_2\end{bmatrix},$$

$$D_1=\begin{bmatrix}0&0&1&0\\0&0&0&1\end{bmatrix}[T_1(h_0)][\overline{w}_n],\ D_2=\begin{bmatrix}0&0&1&0\\0&0&0&1\end{bmatrix}[T_n(h_n)],\ E=-\begin{bmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}[L_k]$$

Inserting Eqn (7) into Eqn (5) and applying inverse Fourier transform, at the same time we use the boundary conditions at the crack faces and the single-valued condition for the displacement components, we obtain the dual integral equation for the present problem:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} m(s,h_k)\{\Delta\tilde{u}_{xk},\Delta\tilde{u}_{yk}\}^T\exp(isx)ds=\{\sigma_1(x),\sigma_2(x)\}^T \tag{8}$$

$$\int_{-\infty}^{\infty}\{\Delta\tilde{u}_{xk},\Delta\tilde{u}_{yk}\}^T\exp(isx)ds=0 \qquad |x|>c \tag{9}$$

where $m(s,h_k)=\begin{bmatrix}0&0&1&0\\0&0&0&1\end{bmatrix}([T_j(y)]([L_{jk}]+[K_{jk}](1-H(j-k))))\begin{bmatrix}0&0&1&0\\0&0&0&1\end{bmatrix}^T$

2.3 Cauchy singular integral equation

Defining dislocation density functions as

$$\eta_1(x)=\frac{\partial}{\partial x}(\Delta u_{xk}),\ \eta_1(x)=\frac{\partial}{\partial x}(\Delta u_{xk}),\qquad |x|\le c \tag{10}$$

Eqn (8) and (9) can be written as

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}s^{-1}m(s,h_0)\int_{-c}^{c}\{\eta_1(u),\eta_1(u)\}^T\exp[is(x-u)]duds=\{\sigma_1(x),\sigma_2(x)\}^T \tag{11}$$

$$\int_{-c}^{c}\{\eta_1(u),\eta_1(u)\}^T du=0 \tag{12}$$

Considering the asymptotic behavior of functions and following methods developed by Erdogan [13], Eqn (11) and (12) can be numerically solved. We can obtain the stress intensity factors at the crack tips.

3 CONCLUSIONS

In this paper, we suggested a new linear multi-layered model for the fracture analysis of FGMs. In this model the FGMs are modeled as a multi-layered medium with the reciprocal of elastic moduli varying linearly in each sub-layer and continuous on the sub-interfaces. Using this model, a FGM strip containing a thorough crack was investigated, and the stress intensity factor was obtained.

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