FORMULATION OF CONTINUOUS/DISCONTINUOUS GALERKIN METHODS FOR STRAIN GRADIENT-DEPENDENT DAMAGE

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ABSTRACT

Continuum damage models are widely used to represent the development of microscopic defects that coalesce into a macroscopic crack. The microscopic defects cause a progressive weakening or softening of the material (damage). Strain gradient-dependent terms have been included in some damage theories to regularize them, and thereby avoid a pathological mesh-dependence in the solution. A strain gradient-dependent damage model is considered here for the simulation of this feature in quasi-brittle materials. In the model considered, the damage parameter depends upon a regularized equivalent strain. The regularization is introduced through a dependency on the Laplacian of an equivalent strain measure. The introduction of the Laplacian of the strain leads to numerical difficulties as the governing differential equations are fourth-order, and additional boundary conditions must be specified. The application of such a model in a standard finite element framework requires C^1 continuity of the shape functions. Here, a continuous/discontinuous mixed Galerkin method is presented which avoids the need for high-order continuity. The formulation allows the use of C^0 or C^{-1} interpolations for the regularized strain field and a C^0 interpolation of the displacement field. Numerical examples are presented to validate the formulation in one and two dimensions. Several interpolations are tested extensively in one dimension in order to provide guidance for the most appropriate formulations in two dimensions. The formulation is applied to crack propagation in a three-point bending test, with the computed result being independent of the discretization.

1 INTRODUCTION

In damage mechanics, a quantity is introduced in the constitutive model which provides a measure of material degradation. Damage degradation can be manifest in a progressive softening of the material. Classical continuum model cannot capture this phenomenon. Regularized continuum models (in particular gradient theories) have been introduced to model softening phenomena. The development of gradient models has however been hindered by the lack of a robust numerical framework. The solution of gradient-dependent continuum problems usually demands a high degree of continuity of the shape functions. Finite element methods with these requirements are expensive in 2D and possibly intractable in 3D. To avoid these problems a reformulation into an implicit gradient form was introduced by Peerlings et al. [1].

Strain gradient-dependent problems can be treated in light of the recent development in discontinuous and continuous/discontinuous Galerkin methods (Arnold et al. [2], Engel et al. [3]). The advantage of this class of methods is the possibility to use C^0 interpolation functions for the displacement, even for continuum theories involving higher-order gradients. Continuity requirements are relaxed and weakly imposed through the addition of weighted residual terms. The formulation is validated through numerical examples in one and two dimensions.

2 FORMULATION OF THE PROBLEM

Consider a body $\Omega \subset \mathbb{R}^d$, with boundary Γ, where d is the spatial dimension. The outward normal to Γ is denoted $\boldsymbol{n}$. The governing equations are:

$$\nabla \cdot \boldsymbol{\sigma} = \boldsymbol{0} \text{ in } \Omega, \tag{1}$$

$$\boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{h} \text{ on } \Gamma_h \tag{2}$$

$$\boldsymbol{u} = \boldsymbol{g} \text{ on } \Gamma_g \tag{3}$$

where ∇ is the gradient operator, $\boldsymbol{\sigma}$ is the stress tensor, $\boldsymbol{h}$ is the prescribed traction on Γ_h and $\boldsymbol{g}$ is the prescribed displacement on Γ_g. The partition of the boundary is such that $\Gamma_h \cup \Gamma_g = \Gamma$ and $\Gamma_h \cap \Gamma_g = \varnothing$.

For the considered damage model, the constitutive equation is given by:

$$\boldsymbol{\sigma} = (1-\omega)\boldsymbol{C} : \nabla^s \boldsymbol{u}, \tag{4}$$

where $\boldsymbol{C}$ is the elasticity tensor, $\nabla^s(\cdot)$ is the symmetric gradient of $(\cdot)$, and ω is the damage parameter, which is a function of a history parameter k, which in turn is related to the regularized equivalent strain measure $\bar{\varepsilon}$ through the Kuhn-Tucker relations. For a strain-gradient dependent damage model, a dependence on strain gradient is introduced (Peerlings et al. [1]):

$$\bar{\varepsilon} = \varepsilon_{eq} + c^2 \Delta \varepsilon_{eq}, \tag{5}$$

where ε_{eq} is an invariant of the local strain tensor, Δ is the Laplacian operator and c is an intrinsic (material) length scale.

The introduction of the second gradient of the deformation in the constitutive relation leads to a fourth-order differential equation in the damaged zone, which implies additional conditions at the interfaces between damaged and undamaged zones. These conditions are provided by continuity requirements if the damaged zone is inside the domain Ω, while additional boundary conditions must be introduced if the damaged zone reaches the external boundary Γ.

3 GALERKIN FORMULATION

Before proceeding with the Galerkin formulation, it is necessary to first establish some definitions. The domain $\overline{\Omega}$ is divided into closed finite elements $\overline{\Omega}_e$ such that:

$$\overline{\Omega} = \bigcup_{e=1}^{nel} \overline{\Omega}_e, \tag{6}$$

where nel is the number of elements. A domain $\tilde{\Omega}$ is also defined such that it does not include element boundaries,

$$\tilde{\Omega} = \bigcup_{e=1}^{nel} \Omega_e. \tag{7}$$

It is also useful to define the interior boundary $\tilde{\Gamma}$,

$$\tilde{\Gamma} = \bigcup_{i=1}^{n_b} \Gamma_i, \tag{8}$$

where Γ_i is the ith interior element boundary and n_b is the number of internal inter-element boundaries.

Before proceeding with the Galerkin formulation of the damage problem, the following finite dimensional function spaces are introduced:

$$\mathcal{S}^h = \{u_i^h \in H^1(\Omega) \mid u_i^h\big|_{\Omega_e} \in P^{k_1}(\Omega_e) \forall e, u_i = g_i \ on \ \Gamma_g\}\,, \tag{9}$$

$$\mathcal{V}^h = \{w_i^h \in H^1(\Omega) \mid w_i^h\big|_{\Omega_e} \in P^{k_1}(\Omega_e) \forall e, w_i = 0 \ on \ \Gamma_g\}\,, \tag{10}$$

$$\mathcal{W}^h = \{q^h \in L^2(\Omega) \mid q^h\big|_{\Omega_e} \in P^{k_2}(\Omega_e) \forall e\}\,. \tag{11}$$

Here, P^k is the space of polynomials of order k, H^1 and L^2 are the usual Hilbert spaces. Note that the spaces $\mathcal{S}^h$ and $\mathcal{V}^h$ correspond to the usual, C^0 continuous finite element shape functions. The space $\mathcal{W}^h$ contains functions which are discontinuous across element boundaries. A continuous/discontinuous Galerkin formulation, which allows the solution of the damage problem using the function space reported in eqns. (9-11), given by Wells et al. [4], is:

find $\boldsymbol{w}^h \in \mathcal{S}^h$ and $\bar{\varepsilon}^h \in \mathcal{W}^h$ such that

$$\begin{aligned} &\int_{\Omega} \nabla \boldsymbol{w}^h : (1-\omega)\, \boldsymbol{C} : \nabla^s \boldsymbol{u}^h d\Omega - \int_{\Gamma_k} \alpha_2 \nabla w_{eq}^h \cdot \boldsymbol{n}\, E c^2 \nabla \varepsilon_{eq}^h \cdot \boldsymbol{n}\; d\Gamma \\ &= \int_{\Gamma_h} \boldsymbol{w}^h \cdot \boldsymbol{h}\, d\Gamma - \int_{\Gamma_k} \alpha_2 \nabla w_{eq}^h \cdot \boldsymbol{n}\, E c^2 h_{\nabla \varepsilon_{eq}}\; d\Gamma \quad \forall \boldsymbol{w}^h \in \mathcal{V}^h, \end{aligned} \tag{12}$$

$$\begin{aligned} &\int_{\Omega} q^h \bar{\varepsilon}^h d\Omega - \int_{\Omega} q^h \varepsilon_{eq}^h d\Omega + \int_{\tilde{\Omega}} \nabla q^h \cdot c^2 \nabla \varepsilon_{eq}^h\; d\Omega - \int_{\Gamma} c^2 q^h \nabla \varepsilon_{eq}^h \cdot \boldsymbol{n}\, d\Gamma - \int_{\tilde{\Gamma}} c^2 [\![q^h]\!] \left\langle \nabla \varepsilon_{eq}^h \right\rangle d\Gamma \\ &- \int_{\tilde{\Gamma}} \left\langle \nabla q^h \right\rangle \cdot c^2 [\![\varepsilon_{eq}^h]\!]\, d\Gamma + \int_{\tilde{\Gamma}} \frac{\alpha_1 c^2}{h_e} [\![q_{eq}^h]\!] \cdot [\![\varepsilon_{eq}^h]\!]\; d\Gamma = 0 \quad \forall q^h \in \mathcal{W}^h, \end{aligned} \tag{13}$$

where α_1 is a penalty-like parameter related to the stabilizing term, α_2 is a penalty term for weakly enforcing a non-standard boundary condition, h_e is a measure of element size, and $[\![\cdot]\!]$ and $\langle \cdot \rangle$ denote the jump and average operator, respectively. Adopting the notation from Arnold et al. [4], the jump operator and the average operator are given by:

$$[\![a]\!] = a_1 \cdot n_1 + a_2 \cdot n_2\,, \tag{14}$$

$$\langle a \rangle = \frac{a_1 + a_2}{2}\,. \tag{15}$$

Eqns (12) and (13) constitute a mixed formulation, with $\bar{\varepsilon}^h$ and $\boldsymbol{u}^h$ being interpolated separately. It can be proven through the application of integration by parts to eqns. (12) and (13) that the proposed weak form is consistent with the original PDE (see Wells et al. [4] for details).

4 NUMERICAL APPLICATIONS

Strain localization in a tensile test on a tapered bar, shown in Figure 1, has been studied with several interpolations (Molari [5]). The commonly used damage law

$$\omega = \begin{cases} 0 \quad if\ k \le k_0 \\ 1 - \dfrac{k_0 (k_c - k)}{k (k_c - k_0)} \quad if\ k_0 \le k \le k_c \\ 1 \quad if\ k \le k_c \end{cases} \tag{16}$$

is considered with k_0=0.0001 and k_c=0.0125. The other data are: Young's modulus $E = 20\times10^4$ MPa, length scale c=1mm. The different elements are called $P^k/P^j(C^i)$ where P^k and P^j are the polynomial of order k and j which interpolate the displacement and the regularized strain measure, respectively. This latter interpolation can be continuous or discontinuous which is indicated by the index i. In the examples shown in the following, $\alpha_1 = 1$.

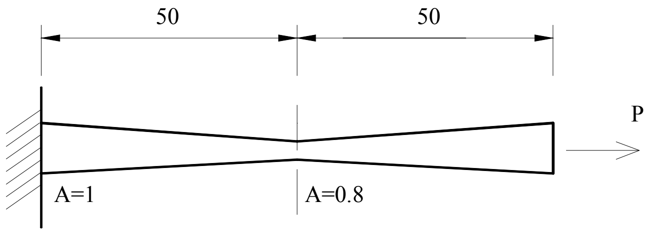
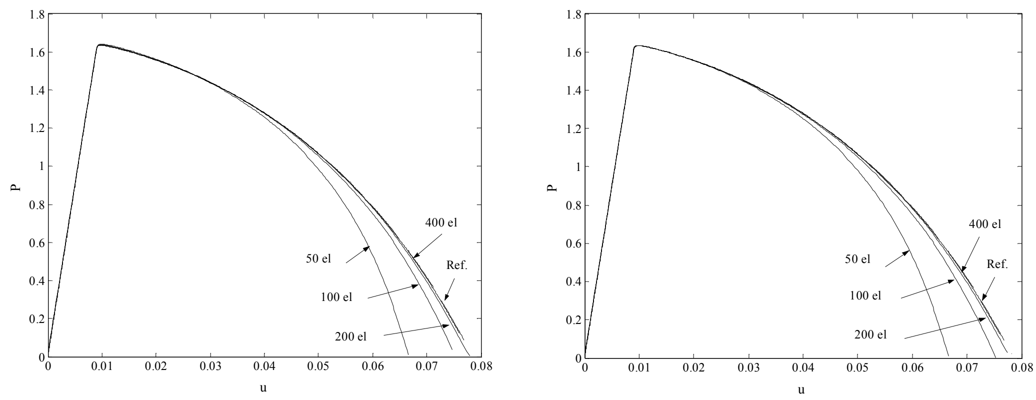


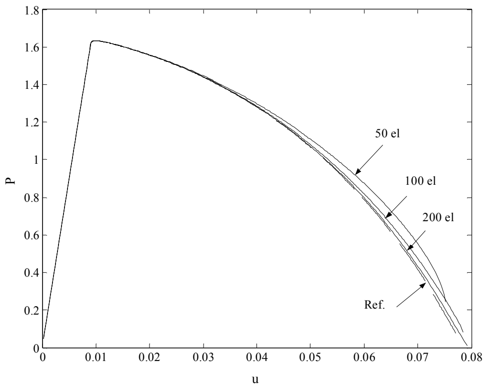
Figure 1: A tapered bar.

In Figure 2, the load-displacement responses for two different discontinuous element types are shown for various discretizations. For both elements, the response converges upon refinement. Figure 3 shows the load-displacement response for continuous elements. Once again, the results converge. In Figures 2 and 3 the reference for comparison is the solution obtained with the P^3/P^2(C^0) interpolation and 200 elements.

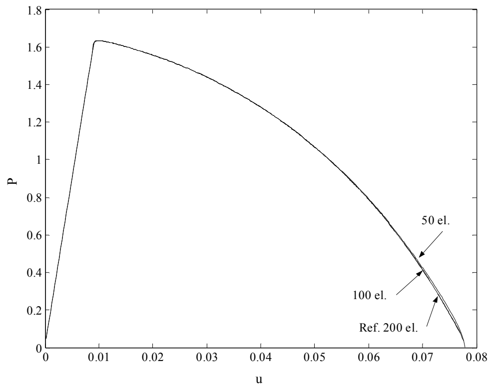


(a) (b)

Figure 2: Load displacement responses for (a) $P^1/P^0(C^{-1})$ and (b) $P^2/P^1(C^{-1})$ elements for various discretizations.



(a)



(b)

Figure 3: Load displacement response for (a) $P^2/P^1(C^0)$ and (b) $P^3/P^2(C^0)$ for various discretizations.

From the one-dimensional analysis, some guidance can be derived as to the most appropriate formulation in two dimensions. It appears that the $P^2/P^1(C^0)$ element provides the best compromise of simplicity and efficiency. It is therefore chosen for further examination in two dimensions, using a triangular element as the base. For the two dimensional tests, a three-point bending specimen is considered (see Figure 4). For this case, the damage law in eqn. (16) is adopted and the following material properties are adopted: Young's modulus $E = 20\times10^4$ MPa, Poisson's ratio ν =0, k_0=0.0001, k_c=0.0125 and c=0.08mm. The equivalent strain measure is defined as:

$$\varepsilon_{eq} = tr(\boldsymbol{\varepsilon}) . \tag{17}$$

While not particularly realistic, this definition makes linearization of the method relatively simple.

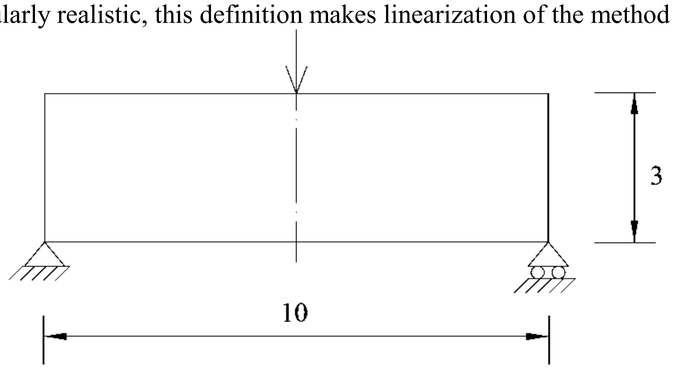


Figure 4: Three-point bending specimen.

The damage contours for two meshes are shown in Figure 5. For both meshes the contours are very close indicating objectivity with respect to the discretization. For this problem the discussion related to the higher-order boundary condition is lengthy. The interested reader is referred to Wells et al. [6].

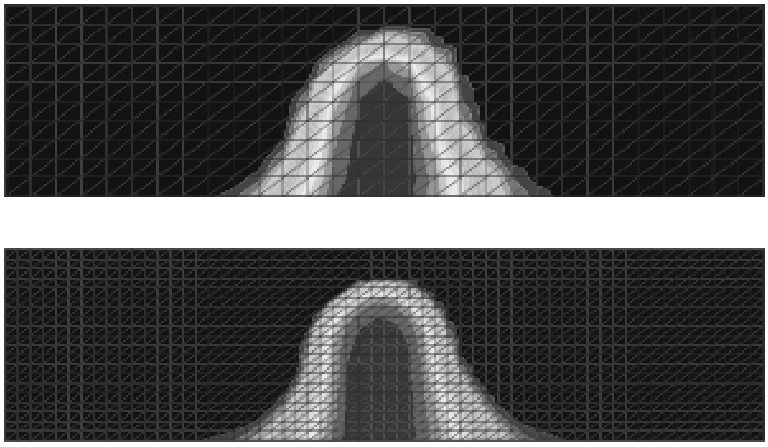


Figure 5: Damage contour for two different meshes.

5 CONCLUSIONS

A mixed continuous/discontinuous Galerkin formulation has been presented for the solution of a strain gradient-dependent damage model. The formulation allows the use of C^0 or even C^{-1} basis functions in situations where classically a C^1 basis functions is required. The formulation was tested in one dimension for various element types and different discretizations. In all cases the formulation converged. The formulation was also tested in two-dimensions using the $P^2/P^1(C^0)$ triangular element, through the simulation of failure in a three-point bending specimen. For the two-dimensional test, the simulation showed no pathological mesh dependency.

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