VECTOR LEVEL SETS FOR CRACKS IN PARTITION OF UNITY METHODS

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ABSTRACT

A new level set method is developed for describing surfaces that are frozen behind a moving front, such as cracks. In this formulation, the level set is described in two dimensions by a three-tuple: the sign of the level set function and the components of the closest point projection to the surface. The vector level set method provides a very simple method for describing the geometry of evolving cracks. In conjunction with local partition of unity methods, such as the Element Free Galerkin or the eXtended Finite Element Method, it enables problems of crack growth to be solved without remeshing.

1 INTRODUCTION

In recent years several methods for the analysis of crack propagation without remeshing have been developed, both in the area of meshless methods [1, 2] and in the framework of the Extended Finite Element Method (X–Fem) [3]. In these approaches, the standard basis is enriched to describe the singular near tip solution and a step function is used for the displacement discontinuity along the crack edges. Even if these methods do not require that the mesh (in the X–Fem) or the nodal arrangement (in the Element Free Galerkin method or other meshless methods) to conform in any way to the crack surface, it is still necessary to describe the geometry of the crack by lines (in 2–D) or surfaces (in 3–D) to locate the discontinuity and to build the discontinuous approximation.

Level sets were developed by Sethian and Osher [4] for problems of interface tracking. The method has evolved significantly since then; a recent summary can be found in Sethian [5]. Burchard et al. [6] and Osher et al. [7] have considered the evolution of curves with level sets. A wide range of applications are presented in [4, 5].

In describing crack surfaces a level set defines the surface of discontinuity by a signed distance function and the evolution of the surface is modeled by a suitable evolution equation. The signed distance function is defined by a set of nodal values, so no explicit representation of the crack is needed. The evolution equation for the surface is usually integrated by finite differences. This approach has been followed in [8] and [9], where the evolution of two dimensional cracks and an elliptic plane crack respectively, has been computed, and in [10, 11] for the general case of 3–D crack evolution.

The standard continuous polynomial basis for the displacement is enriched with discontinuous and near tip fields through a local partition of unity (PU) [12]. This was first applied to fracture in Belytschko and Black [13], where the asymptotic nearfield for a crack was incorporated by a local PU and the discontinuity in this field was used to represent the crack discontinuity independent of the mesh. By adding a step function, Moës et al. [3] and Dolbow et al. [14] generalized the above so that arbitrary crack growth could be modelled without remeshing. This advance led to a methodology to model arbitrary discontinuities independent of the mesh, Belytschko et al. [15].

Partition of Unity methods form a natural combination with level sets. For this combination, only nodal data are required to describe the crack and the finite element enrichment. XFEM and level set were first applied to crack problems in Stolarska et al. [8]; here two orthogonal level sets were used, one for the crack surface, the second to locate the cracktip. The level sets were updated by the fast marching method. In Moës et al. [10] and Gravouil et al. [11] the method was extended

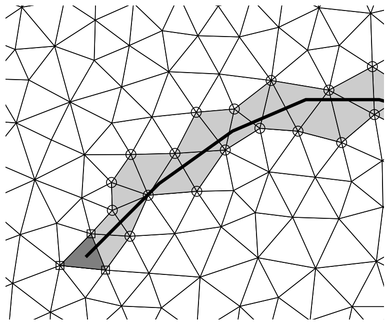

Figure 1. An arbitrary crack in a triangular mesh and the sets $\mathcal{N}_{\rm cr}$ (circles) and $\mathcal{N}_{\rm tip}$ (squares).

to three dimensions. The level sets were updated by solving hyperbolic PDE's; special techniques were developed to freeze the crack surface.

One drawback of the preceding methods is that the algorithm is rendered more complicated by the need to "freeze" the level set describing the existing crack surface. In some ways, in fact, standard level set methods are not ideal for cracks since they were intended for moving interfaces. The growth of a crack in three dimensions can perhaps be better viewed as the evolution of a surface by the evolution of a curve (or the motion of a point in the two dimensions). However, the path of the curve (or point) must be remembered (i.e. stored), for it constitutes the crack surface. Thus an ideal method would provide the evolution of the curve (or point) and describe the crack it generates in a simple form.

In Ventura et al. [16] a vector level set method was developed in the context of meshless methods. The technique has been improved and applied to finite elements in [17]. Vector level sets describe the location of a surface by a four-tuple in three dimensions, a three-tuple in two dimensions: the sign of the level set and the components of the closest point projection vector. The update of the vector level set field involves only a few geometric equations, so in contrast to standard level set methods, no partial differential equations need be solved to update the vector level sets. Vector level set methods have been used for propagating surfaces by Steinhoff et al. [18] and Ruuth et al. [19].

In the present paper the vector level set methodology for cracks is presented for partition of unity methods, both in mesh free and finite element discretizations. Peculiarities and analogies in the two cases are highlighted, and some simulation results will be commented in the full length paper.

2 MODELING CRACKS BY PARTITION OF UNITY

Partition of unity methods allow for a simple treatment of cracks introducing discontinuities at given positions and enriching the usual polynomial basis for obtaining a better approximation. In the following we will denote by domain of influence of a node the domain where the corresponding shape function(s) are nonzero. This is the support of the node weight function in Moving Least Squares (Element Free) approximations, and the domain of the elements connected to the node in finite element approximations.

Let $\mathcal{N}_{\rm cr}$ the set of the nodes whose domain of influence is intersected by the crack surface and does not contain the crack tip, and let $\mathcal{N}_{\rm tip}$ the set of nodes whose domain of influence contain the crack tip.

In Figure 1 this concept is shown with reference to finite elements.

The crack discontinuity is represented in partition of unity methods by enriching the standard approximation space with the Heaviside step functions for nodes in the set $\mathcal{N}_{\rm cr}$, and with the Westergaard's solution for nodes in the set $\mathcal{N}_{\rm tip}$. The following expression for the discretized displacement field is considered:

$$\mathbf{u}^h(\mathbf{x}) = \sum_{I\in\mathcal{N}} N_I(\mathbf{x})\mathbf{u}_I + \sum_{I\in\mathcal{N}_{\rm cr}} N_I(\mathbf{x})H\big(f(\mathbf{x})\big)\mathbf{a}_I + \sum_{I\in\mathcal{N}_{\rm tip}} N_I(\mathbf{x})\sum_{l=1}^{4} B^{(l)}(r,\theta)\mathbf{b}_I^{(l)} \quad (1)$$

where $N_I(\mathbf{x})$ are shape functions, $\mathbf{u}_I$, $\mathbf{a}_I$ and $\mathbf{b}_I^{(l)}$ are the displacement and enrichment nodal variables, f is the signed distance from the crack line, H is the modified Heaviside function ($H = 1$ for any $f > 0$, H$= -1$ for any $f < 0$) and $B^{(l)}(r,\theta)$ is a basis that approximately spans the Westergaard solution for the crack tip field:

$$\mathbf{B}(r,\theta) = \left(\sqrt{r}\sin\frac{\theta}{2},\ \sqrt{r}\cos\frac{\theta}{2},\ \sqrt{r}\sin\frac{\theta}{2}\sin\theta,\ \sqrt{r}\cos\frac{\theta}{2}\sin\theta\right) \quad (2)$$

where r is the distance from the crack tip, while a discontinuity is introduced along $\theta = \pm\pi$. The signed distance function plays a central role in the evaluation of the enriched displacement field (1). In fact, the enrichment function H is the sign of the distance, while f and r are used for determining the polar angle θ. The actual geometric surface of the crack is not needed in evaluating the enrichment when the signed distance function and the position of the crack tips are known. The standard level set approach to describing cracks defines the crack as the zero level contour of the signed distance function f. The crack tip position will be part of the crack data. Alternatively it will be determined by considering the intersection between the zero level contour and a second orthogonal level set function.

3 VECTOR LEVEL SET

The location of the crack is described in $\mathbb{R}^2$ by a three-tuple consisting of the sign of the distance and the components of the closest point projection to the surface. In addition the locations of the crack tips are stored. Given the position of a crack tip, we use the mechanical model to compute, based on a fracture criterion, the advance vector $\mathbf{s}$ determining the new position of the crack tip. These data allow for evaluating geometrically the signed distance and store it at the enriched nodes.

We assume that the level set function f has the structure of a compound object $\tilde{f}$, given by a vector $\mathbf{f}$ and a Boolean H having values ± 1:

$$\tilde{f}(\mathbf{x},t) = \{\mathbf{f}(\mathbf{x},t),\ H(\mathbf{x},t)\} \quad (3)$$

At a fixed time t, $\mathbf{f}(\mathbf{x},t)$ is the vector joining the point $\mathbf{x}$ and its closest point projection on the crack line $\Gamma_{\rm cr}$. It is oriented so it points from $\mathbf{x}$ to the crack line, Fig. 3, i.e. $\mathbf{f} = \bar{\mathbf{x}} - \mathbf{x}$ where $\bar{\mathbf{x}}$ is the closest point projection of $\mathbf{x}$ onto $\Gamma_{\rm cr}$. The signed distance function $f(\mathbf{x},t)$ can be computed from $\tilde{f}(\mathbf{x},t)$ through the formula $f(\mathbf{x},t) = \|\mathbf{f}(\mathbf{x},t)\|\, H(\mathbf{x},t)$.

3.1 Level set evolution and computation

Let $\mathcal{F}^n$ be the set of level set points where the function $\tilde{f}$ is defined at the $n^{\rm th}$ step of the crack evolution. At this step, given $\mathcal{F}^{n-1}$ and the crack tip advance vector $\mathbf{s}^n$ (the secant to the next crack segment), let $\mathcal{F}_a$ be the set of points whose signed distance is to be computed. This may be specialized differently for finite elements and mesh free methods, according to the data required to compute the stiffness matrix in the two cases. In moving least squares approximations the body domain is usually subdivided into quadrature cells, so that level set data are required at all those nodes whose domain of influence includes the quadrature points. In this way a constant, fixed bandwidth, can be chosen, suitably larger than the maximum support radius, as illustrated in Fig. 3(a). Considering that the crack propagates of a small amount, a circular search area can be employed as well [16]. When finite elements are used, the enriched nodes are the ones of the elements intersected by the crack, so that their number is usually significantly smaller compared to mesh free methods and the level set domain is immediately determined, Fig. 3(b).

As the crack advances by the segment $\mathbf{s}^n$, new points will have a geometric closest point projection onto the segment. The shaded areas represent the subsets where the level set points are located. From Figure 3(a) three different cases are possible:

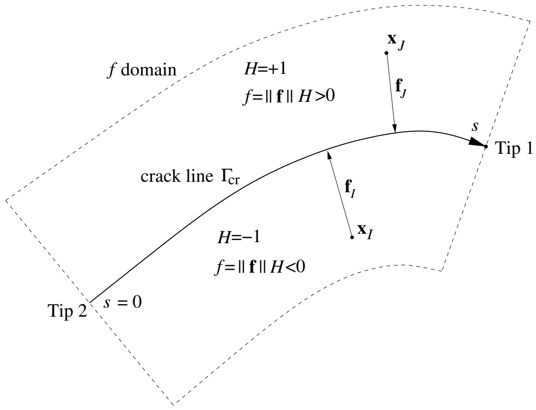

Figure 2. Definition of $\tilde{f}$ at two nodal points.

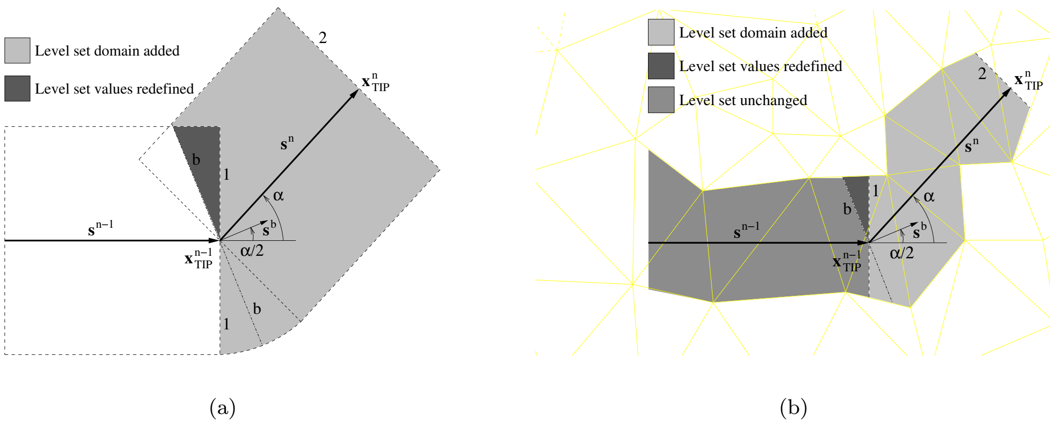

Figure 3. Representation of the set $\mathcal{F}_a$: (a) for element free methods; (b) for finite elements.

- the closest point projection is an orthogonal projection on the segment $\mathbf{s}^n$;
- the closest point projection redefines the values of some level set points;
- the closest point projection is given by the distance to the tip position $\mathbf{x}_{TIP}^{n-1}$ at the previous advance step.

With the aforementioned differences between element free and finite element discretizations, from Fig. 3(a), the set $\mathcal{F}_a$ where the level set function is to be evaluated is given by the points in the domain $\mathcal{E}_g^n$, lying before segment 2 and ahead segments 1 or b, where b bisects the angle between $\mathbf{s}^{n-1}$ and $\mathbf{s}^n$.

Once the domain of the level set is determined, the closest point projection and the sign of the distance are determined by simple vector algebra at the discretization nodes, where they are stored as well.

3.2 Extrapolation/Interpolation of the vector level set

To evaluate the stiffness matrix, level set values are required at the quadrature points. The vector representation of the closest point projection allows for a straightforward evaluation of the level set function at any point in the domain around the crack.

Several alternatives have been tested, and results will be given in the full paper. Here we mention vector extrapolation [16], shape functions [11] and circle interpolation [17].

4 CONCLUSIONS

Partition of Unity methods and level sets form a very effective combination for modelling crack growth without remeshing. In the paper the accuracy of the vector level set method will be discussed compared to classical geometric crack representation and some applications will be shown.

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