SIZE EFFECTS IN GRAINED MATERIALS: EXTREME VALUE THEORY APPROACH

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ABSTRACT

The present paper provides a statistical model to the size effect on grained materials tensile strength and fracture energy. It has been already demonstrated by using extreme value theory (Carpinteri et al. [1]) that the scaling law obtained for the tensile strength introducing a doubly truncated distribution of flaws, under the hypothesis of Weibull's weakest link, resembles the Multi-Fractal Scaling Law (MFSL), already proposed by the first Author through fractal concepts. A recent improvement of the model has been proposed (Carpinteri et al. [2]) observing that the weakest link in grained materials is usually represented by the interface between the matrix and the grains. Thus, the flaw distribution can be represented by the grain size distribution, expressed as a probability density function (PDF) of the grain diameters, rather than by an arbitrary flaw distribution. In this work, introducing the size-independent fractal cohesive model and considering micromechanical models for the critical displacement w_c, we draw a link also between the fracture energy and the largest aggregate grain inside the specimen and compute the fracture energy as a function of the specimen size. The obtained scaling law is again in substantial agreement with the MFSL for the fracture energy proposed by the first Author. A further result provided by the proposed approach is the description of the scatter increase of both tensile strength and fracture energy values when testing small specimens. This trend is confirmed by experimental data available in the literature.

1 INTRODUCTION

With size effect we mean the dependence of one or more material parameters on the size of the material specimen. It is easy to realize the importance of this topic in engineering design. Recently, the scientific community dedicated significant efforts in order to have a consistent description of this phenomenon and to highlight the physical mechanisms that lie behind it. Dealing specifically with concrete structures, it was seen that tensile strength decreases with the structural size, whereas fracture energy increases (Carpinteri [3,4]). In other words, the larger is the structure, the more brittle the structural behaviour results to be.

Aim of the present paper is to develop a statistical model providing the PDF of grained materials tensile strength and fracture energy for specimens of different sizes. Since the interface between the matrix and the aggregates is the weakest link, we assume that the PDF of the flaw sizes can be realistically represented by the PDF of grain diameters (Carpinteri et al. [2]). Our analysis will therefore start with the description of the aggregate grading inside a grained material.

2 STEREOLOGICAL ANALYSIS OF THE GRAIN SIZE DISTRIBUTION FUNCTION

The basis for the dimensional characterization of the aggregate is the sieve analysis. The sieve curve describes the weight fraction W(d) of the aggregate passing through a sieve with d-wide mesh. Due to its good packing properties, the most common sieve curve used to prepare concrete is the so-called Füller curve:

$$W(d) = \sqrt{\frac{d}{\phi_{\max}}} \tag{1}$$

Assuming that the aggregates are spheres with diameter d comprised between $\phi_{\min}$ and $\phi_{\max}$, it can be easily shown (Stroeven [5]) that the Füller sieve curve of eqn (1) can be expressed in terms of grain size distribution function as follows:

$$f_d(d) = \frac{2.5}{1-\alpha^{-2.5}} \frac{\phi_{\min}^{2.5}}{d^{3.5}} \tag{2}$$

where α = $\phi_{\max}/\phi_{\min}$ and $f_d(d)$ is a probability density function (PDF). Note that the first denominator in the previous expression is very close to the unity; nevertheless, differently from other approaches (Carpinteri et al. [6]), we cannot neglect it in the following computation since that term will be raised to very high exponents.

In order to link the concrete volume with the number of grains inside it, we need one more parameter, i.e., the volume percentage f_a of the aggregates. The total number of particles inside a volume V is therefore obtained on average, dividing the total volume of aggregates inside the concrete volume by the average grain volume:

$$N = \frac{f_a V}{\frac{\pi}{6}\overline{d^3}} \tag{3}$$

where $\overline{d^3}$ is the third moment of the PDF described in eqn (2).

Following the procedure outlined by Carpinteri et al. [1], we compute the expression of the PDF of the maximum diameter of the N aggregate particles contained within a given volume V, defined as: $d_{\max} = \max\{d_1, d_2, \ldots, d_N\}$. Starting from the hypothesis that the N aggregate diameters are i.i.d. variables (independent and identically distributed), the extreme value theory provides the PDF of $d_{\max}$:

$$f_{d_{\max}}(d) = N\left[F_d(d)\right]^{N-1} f_d(d) \tag{4}$$

3 TENSILE STRENGTH

Now we derive the relation between the strength and the grain number, i.e. the structural size (eqn 3). As stated in the introduction, we assume that the strength depends on the largest flaw according to Weibull's weakest link hypothesis. Furthermore, we assume that defect interactions are negligible and, due to interface weakness, we represent the effect of a spherical particle as that of a penny-shaped crack with the same diameter. Hence we can write the ultimate tensile strength as:

$$\sigma_u(d_{\max}) = \frac{\pi}{\sqrt{2}} \frac{K_{IC}}{\sqrt{\pi d_{\max}}} \tag{5}$$

Eqn (5) states that the tensile strength decreases along with the inverse of the square root of the largest grain diameter. The minimum strength is achieved when $d_{\max}$ is equal to $\phi_{\max}$ and is denoted by f_t. Thus eqn (5) can be rewritten in nondimensional form:

$$\sigma_u(d_{\max}) = f_t \sqrt{\frac{\phi_{\max}}{d_{\max}}} \tag{6}$$

From the previous equation, it is clear that σ_u is a statistical variable as long as $d_{\max}$. The PDF of σ_u depends on the PDF of $d_{\max}$ according to the following relationship:

$$f_{\sigma_u}(\sigma_u) = f_{d_{\max}}(d) \left|\frac{\mathrm{d}\, d_{\max}}{\mathrm{d}\, \sigma_u}\right| = N \frac{\left[1-\left(\sigma_u / f_t\sqrt{\alpha}\right)^5\right]^{N-1}}{\left(1-\alpha^{-2.5}\right)^{N-1}} \left(\frac{5}{f_t^5 \alpha^{2.5}}\right)\sigma_u^4 \tag{7}$$

Computing the mean value of the PDF given by eqn (7) we obtain the average tensile strength, as a function of the number N of grains. In nondimensional form, its final expression is given by:

$$\frac{\overline{\sigma_u}}{f_t} = 1 + \frac{\sqrt{\alpha}}{5(1-\beta)^N} B_{(\beta,1)}\left(\frac{1}{5}, N+1\right) \tag{8}$$

where $\beta = \alpha^{-2.5}$ and

$$B_{(a,b)}(n,m) = \int_a^b (1-x)^{m-1} x^{n-1}\, \mathrm{d}x \tag{9}$$

is the Generalized Incomplete Beta Function. Eqn (9) allows to compute the mean tensile strength as a function of the particles number N and of the parameter α. Equivalently, by using eqn (3), it is possible to highlight the size-scale effect with respect to the nondimensional structural size (b/t), distinguishing the case of two- and three-dimensional scaling. A first important remark is that only the ratio α of maximum to minimum aggregate size plays a role, whilst the value of the maximum diameter does not affect the function shape. Results are summarized in Fig. 1(a), where the log-log plot evidences that both the curves exhibit a similar behaviour, with two ranges. In the lowest one the curves decrease with a constant slope, equal to 0.4 and 0.6 for two- and three-dimensional scaling, respectively. At the larger scales they present an asymptotic trend towards the unity. From a mechanical point of view, this yields an average tensile strength approaching f_t for sufficiently large sizes. For a comparison of the proposed scaling law with experimental data the interested reader is referred to Carpinteri et al. [2].

4 FRACTURE ENERGY

In order to obtain the relationship between the fracture energy and the grain number, we start analyzing the effect of the largest grain diameter upon the critical displacement w_c. The long tail usually shown by the cohesive laws of grained materials is due to the bridging action between the crack lips exerted by the grains. The larger are the grains, the larger is the distance between the lips at which the interaction vanishes. The final part of the softening regime is strictly related to the pull out of the largest grains. When a grain is being pulled out from the matrix, interlocking between the grain and the matrix supplies the resistance to the separation of the plane. Of course, unlike fibre pull out, where the critical separation of the failure plane is just equal to half the length of the fibre, the critical distance in grain pull out is much smaller than the aggregate radius, as shown by several experiments.

To carry on our analysis, we do not need the exact value of the critical distance: it is sufficient to know how it varies along with the grain diameter. Although different hypotheses can be formulated, we will assume that the critical displacement is proportional to the diameter of the largest grain upon fracture surface, since it is the last one to be pulled out:

$$w_c = k_1 d_{\max} \tag{10}$$

k_1 being a material constant. Note that, in analogy with eqn (10), eqn (6) for the tensile strength can be rewritten as $\sigma_u = k_2 / \sqrt{d_{\max}}$. Again, k_2 is a material constant.

Assuming that the shape of the cohesive law is size independent, the dependence of the fracture energy with respect to $d_{\max}$ is straightforward. In fact, indicating by f the size independent function describing the dimensionless cohesive law yields:

$$\sigma/\sigma_u = f(w/w_c) \tag{11}$$

Eqn (11) is equivalent to state that the dependence of the cohesive law on the structural size is only due to the size dependence of its peak (the tensile strength) and its tail (the critical displacement). Observe that this statement is also implicit in the fractal cohesive crack model presented by Carpinteri et al. [7]. According to its definition, we can compute the fracture energy from Eq. (11):

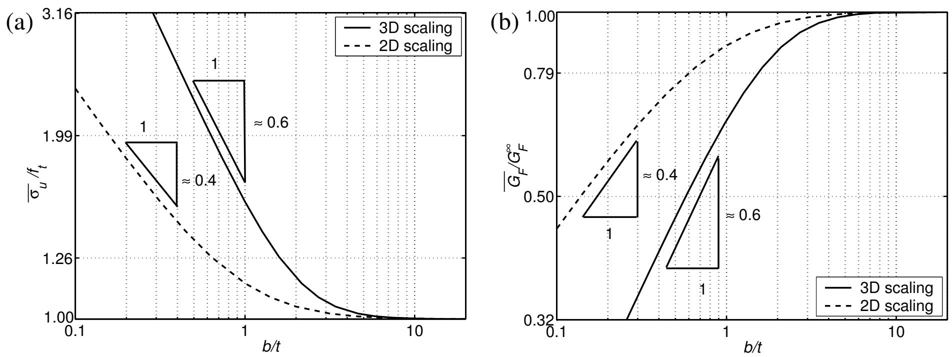

Figure 1: Size effect on tensile strength (a) and fracture energy (b)

$$\mathcal{G}_{\mathcal{F}} = \int_0^{w_c} \sigma(w)\,\mathrm{d}w = w_c\,\sigma_u \int_0^1 f\left(\frac{w}{w_c}\right) \mathrm{d}\left(\frac{w}{w_c}\right) = w_c\,\sigma_u\,g_F = k_1\,k_2\,g_F\sqrt{d_{\max}} \tag{12}$$

where g_F is the value of the integral: it is a dimensionless constant depending on the shape of the cohesive law (e.g. equal to 1/2 for a linear cohesive law). Eqn (12) provides the dependence of the fracture energy upon the largest grain diameter $d_{\max}$ we were looking for. The same dependence of $\mathcal{G}_{\mathcal{F}}$ upon the largest grain diameter has been proposed on experimental evidence by several authors (Wolinski et al. [8], Li et al. [9]), so that the hypothesis stated in eqn (10) is confirmed. Noting that in the limit of large specimen size $d_{\max}$ tends to $\phi_{\max}$, we can evaluate the ratio of fracture energy $\mathcal{G}_{\mathcal{F}}$ at a generic size to fracture energy $\mathcal{G}_{\mathcal{F}}^{\infty}$ for structural size tending to infinity as:

$$\mathcal{G}_{\mathcal{F}}(d_{\max}) = \mathcal{G}_{\mathcal{F}}^{\infty}\sqrt{\frac{d_{\max}}{\phi_{\max}}} \tag{13}$$

From eqn (13) it is clear that the fracture energy $\mathcal{G}_{\mathcal{F}}$ is a statistical variable as well as $d_{\max}$ (and σ_u). Thus, to evaluate the average fracture energy as a function of the particle number N, we should follow a similar procedure to that previously described for the ultimate tensile strength. As a result, we obtain the following expression for the nondimensional fracture energy:

$$\frac{\overline{\mathcal{G}_{\mathcal{F}}}}{\mathcal{G}_{\mathcal{F}}^{\infty}} = 1 - \frac{1}{5\sqrt{\alpha}(1-\beta)^N} B_{(\beta,1)}\left(-\frac{1}{5}, N+1\right) \tag{14}$$

The nondimensional mean value of the fracture energy can be calculated by eqn (14) as a function of the particles number N and of the parameter α. As stated for the ultimate tensile strength, only the ratio α between maximum and minimum aggregate size plays a role, whilst the value of the maximum diameter does not affect the function shape. Results are summarized in Fig. 1(b) where the mean fracture energy is plotted vs. structural size (b/t) for two- and three-dimensional scaling: both the curves exhibit a similar behavior. At the smaller scales, the curves increase with a constant slope, approximately equal to 0.4 and 0.6 for two- and three-dimensional scaling, respectively, whilst at the larger scales they present an asymptotic trend towards the unity.

To highlight the true size-scale effect on tensile strength and fracture energy, we linked the number N with the considered volume (see eqn (3)), by specifying the volume V in which the largest aggregate should be sought. Considering now the size effect of the fracture energy, it could be argued that the relation between the grain number and the structural size differs from the one used for the tensile strength, since the largest grain must be sought on the fracture surface and not inside a given volume V. Nevertheless, the fact that fracture starts from the grain with the largest diameter allows us to assert that the largest grain inside V will belong to the fracture surface.

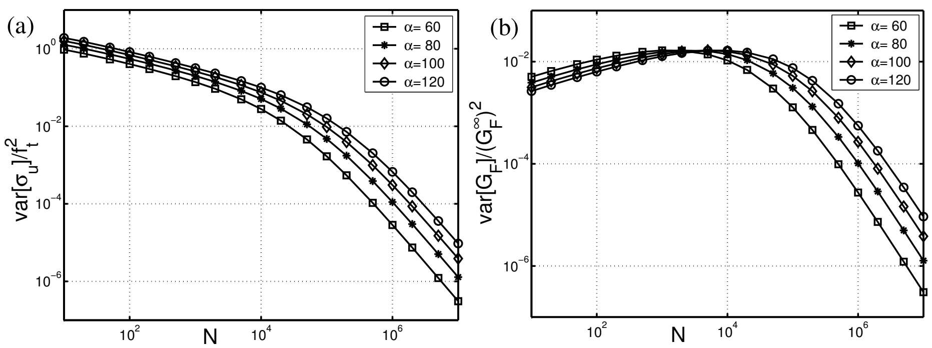

Figure 2: Nondimensional variance of the tensile strength (a) and of the fracture energy (b)

5 INCREASE IN THE STATISTICAL DISPERSION AT THE SMALLER SCALES

The first moments of the PDFs of the tensile strength and of the fracture energy provide the mean values of these quantities. As shown in the previous section, interesting considerations about their size effect can be drawn. On the other hand, also the higher order moments of the PDFs provide useful information. The second moments represents the variances of the PDFs. They contain information about the scattering of the values for different N values, i.e. varying the structural size. In fact, computing the variances of both the tensile strength and the fracture energy, we obtain:

$$\frac{\mathrm{var}[\sigma_u]}{f_t^2} = 1 + \frac{2\alpha}{5(1-\beta)^N} B_{(\beta,1)}\left(\frac{2}{5}, N+1\right) - \left[1 + \frac{\sqrt{\alpha}}{5(1-\beta)^N} B_{(\beta,1)}\left(\frac{1}{5}, N+1\right)\right]^2 \tag{15}$$

$$\frac{\mathrm{var}[\mathcal{G}_\mathcal{F}]}{\left(\mathcal{G}_\mathcal{F}^\infty\right)^2} = 1 - \frac{2}{5\,\alpha(1-\beta)^N} B_{(\beta,1)}\left(-\frac{2}{5}, N+1\right) - \left[1 + \frac{1}{5\sqrt{\alpha}(1-\beta)^N} B_{(\beta,1)}\left(-\frac{1}{5}, N+1\right)\right]^2 \tag{16}$$

for the tensile strength and for the fracture energy, respectively.
The nondimensional variances are plotted in Figs. 2(a) and 2(b) vs. the grain number for different values of the ratio. As can be seen, the variances increase diminishing the number of grains, i.e. as the structural size decreases. This means that the present model predicts not only a variation of the tensile strength and fracture energy values when testing specimens of different sizes, but also a wider scatter of the measured data for small sizes. Both these trends are confirmed by several experimental results (see, for instance, [10]). Furthermore, note that, in the bi-logarithmic plots in Figs 2(a) and 2(b), the slope of the curves for large N values is the same.

More information is provided by the graphs in Fig. 3, where the dispersion bands within 33% give a clear indication of the asymmetry of the two PDFs given by eqns (8) and (14) as the structural scale changes.

6 CONCLUSIONS

In this paper, we presented a statistical approach to the size scale effects on strength and toughness of grained materials. A first important result of the present analysis is the size effect predicted, which coincides with the one provided by the MFSLs, thus confirming by another way the soundness of the fractal approach to size effect in quasi-brittle materials as proposed by Carpinteri and co-workers [3,4,7].

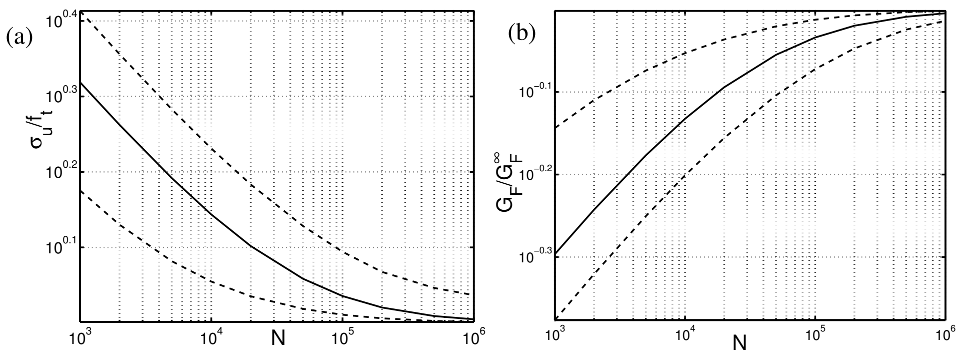

Figure 3: Dispersion bands within 33% probability for tensile strength (a) and fracture energy (b)

A second important result refers to the increase of the statistical dispersion at the smaller scales. This trend agrees with results from experimental investigations. Therefore, the larger scatter when testing small specimens should be carefully considered, especially if the final goal is to extrapolate strength and fracture energy values to full-size structures.

7 ACKNOWLEDGEMENTS

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8 REFERENCES

[1] Carpinteri, A., Ferro, G. & Invernizzi, S.: The nominal tensile strength of disordered materials: a statistical fracture mechanics approach. *Engineering Fracture Mechanics*, 58: 421-435 (1997).

[2] Carpinteri, A., Cornetti, P. & Puzzi, S.: A stereological analysis of aggregate grading and size effect on concrete tensile strength. *International Journal of Fracture*, in press.

[3] Carpinteri A.: Fractal nature of material microstructure and size effects on apparent mechanical properties. *Mechanics of Materials*, 18: 89-101 (1994).

[4] Carpinteri A.: Scaling laws and renormalization groups for strength and toughness of disordered materials. *International Journal of Solids and Structures*, 31: 291-302 (1994).

[5] Stroeven, P.: A stereological approach to roughness of fracture surfaces and tortuosity of transport paths in concrete. *Cement and Concrete Composites*, 22: 331-341 (2000).

[6] Carpinteri, A. & Cornetti, P.: Size effects on concrete tensile fracture properties: an interpretation of the fractal approach based on the aggregate grading. *Journal of the Mechanical Behaviour of Materials*, 13: 233-246 (2002).

[7] Carpinteri, A., Chiaia, B. & Cornetti, P.: A scale-invariant cohesive crack model for quasi-brittle materials. *Engineering Fracture Mechanics*, 69: 207-217 (2002).

[8] Wolinski, S., Hordijk, D.A., Reinhardt, H.W. & Cornelissen, H.A.W.: Influence of aggregate size on fracture mechanics of parameters of concrete. *International Journal of Cement Composites and Lightweight Concrete*, 9: 95–103 (1987).

[9] Li, Q., Duan, Y. & Wang, G.: Behaviour of large concrete specimens in uniaxial tension. *Magazine of Concrete Research*, 54: 385–391 (2002).

[10] Karihaloo, B.L., Abdalla, H.M. & Xiao, Q.Z.: Size effect in concrete beams. *Engineering Fracture Mechanics*, 70: 979–993 (2003)