ON COUPLING ANISOTROPIC DAMAGE AND INTERNAL FRICTION IN MODELING BRITTLE MATERIALS

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ABSTRACT

An internal friction-damage coupled model is derived from a micromechanically based description of the response of brittle materials. The material is modeled as an elastic isotropic matrix containing a statistically uniform distribution of crack-like defects. Under the simplifying assumption of non-interacting and self-similar propagating flat cracks having the same shape and random locations and orientations, displacement discontinuities due to microcrack opening and sliding are treated as inelastic contributions to the mean strain.

The model is based on two tensor-valued internal variables, representing damage and frictional contact tractions and governing the microscopic mechanisms responsible for the global inelastic response. The use of a tensor-valued variable for damage, in particular, makes the model to be capable of describing the load-induced anisotropy of damage in brittle materials. In the framework of thermodynamics with internal variables, overall frictional sliding and crack growth criteria with associated flow rules are introduced to complement the model that should be formulated in incremental form, because of nonlinearity and stress-path-dependence of the constitutive response to arbitrary stress applied to a brittle material at an arbitrary current state.

As an example, the model response to proportional loading has been analysed. In this case no crack changes its status and explicit solutions are possible. On the basis of these results, biaxial and triaxial failure envelopes, together with some characteristic stress-strain curves, have been obtained and used for both identification and validation of the model.

1 INTRODUCTION

Brittle behavior exhibited by many real materials is the result of nucleation and propagation of crack-like microdefects, commonly referred to as damage. Most literature has focused on modeling phenomenologically and micromechanically the progressive degradation of mechanical properties for such materials due to damage evolution (e.g. Krajcinovic [1,2], Carol [3], Feng [4]). On the other hand, under predominant compressive stress states, damage may be accompanied by residual deformation due to possible frictional resistance to sliding of the internal crack surfaces. Despite the fact that damage-elasto-plasticity theories have been formally addressed and discussed [2], the frictional sliding-damage interaction has been often disregarded. Restricting the analysis to plane loading conditions, models for brittle solids under compression based on the sliding crack mechanism have been a first step (Nemat-Nasser [5], Basista [6]).

Within the framework of thermodynamics with internal variables, convenient for dealing with the constitutive modeling of such dissipative materials, Gambarotta and Lagomarsino [7] incorporated the damage – frictional sliding interaction in their microcrack damage model, based on the concept of damage planes. The ensemble of microcracks was thought of as consisting of sets of identical equi-oriented flat cracks and the orientation fields of the characteristic size of the cracks and of the vector of contact tractions were treated as internal variables. Built on this previous work, the strategy applied in this paper for the reduction of the computational effort uses approximate representations of the orientation fields mentioned above in terms of appropriate series expansion of generalized spherical harmonics. A more convenient description for damage and contact tractions in terms of two second-order tensors as new overall internal variables is then obtained. A first simple version of this approach has been recently proposed by Gambarotta [8] and is developed here as a generalization of the isotropic damage model of Brencich and Gambarotta [9], who adopted a single scalar variable to describe damage.

2 THE CONSTITUTIVE EQUATION

Brittle materials are modeled as elastic isotropic matrices containing a population of non-interacting and self-similarly propagating flat cracks having the same shape and random locations and orientations. Each set of equi-oriented cracks is described in terms of the unit vector $\mathbf{n}\in\Omega$ normal to the crack plane (the unit hemi-sphere in $\Re^3$ being denoted by Ω) and of the average characteristic size of the cracks. Normal and tangential displacement jumps across the crack faces are treated as inelastic contributions $\mathbf{E}_n$ and $\mathbf{E}_t$ to the mean strain $\mathbf{E}$ that can then be written as:

$$\mathbf{E}=\mathbb{K}\mathbf{T}+\mathbf{E}_n+\mathbf{E}_t\,, \tag{1}$$

together with:

$$\mathbf{E}_n=\frac{c_n}{2\pi}\int_\Omega \alpha_n^3(\sigma_n-p_n)\,\mathbf{n}\otimes\mathbf{n}\;\mathrm{d}\Omega \quad\text{and}\quad \mathbf{E}_t=\frac{c_t}{2\pi}\int_\Omega \alpha_n^3(\boldsymbol{\tau}_n-\mathbf{f}_n)\odot\mathbf{n}\;\mathrm{d}\Omega\,, \tag{2}$$

where $\mathbb{K}$ is the fourth-order elastic compliance tensor of the matrix and $\mathbf{T}$ is the mean stress. Operators $\otimes$ and $\odot$ in eqn (2) denote diadyc product and its symmetric part, respectively; whereas dΩ is the infinitesimal solid angle representative of the neighborhood of $\mathbf{n}$. For each set of $\mathbf{n}$-oriented cracks: $\alpha_n\geq 0$ is the ratio between the current average crack size and a reference size, assumed as damage variable related to the $\mathbf{n}$-oriented plane; $\sigma_n=\mathbf{n}\bullet\mathbf{T}\mathbf{n}$ and $\boldsymbol{\tau}_n=(\mathbf{I}-\mathbf{n}\otimes\mathbf{n})\mathbf{T}\mathbf{n}=(\mathbf{I}-\mathbf{n}\otimes\mathbf{n})\mathbf{T}'\mathbf{n}$ are the resolved stresses on the crack plane, the deviatoric component of $\mathbf{T}$ being denoted by $\mathbf{T}'$ (tr$\mathbf{T}'$=0); p_n and $\mathbf{f}_n$ are, respectively, the normal and tangential contact tractions acting on the crack faces and assumed to be uniform. Finally, under the further assumption of identical crack number density for all the planes, constants c_n and c_t are, respectively, normal and tangential microcrack compliance parameters.

According to this approach, radially symmetric scalar-valued functions $\alpha_n=\alpha(\mathbf{n})=\alpha(-\mathbf{n})$ and $p_n=p(\mathbf{n})=p(-\mathbf{n})$ and anti-symmetric vector-valued function $\mathbf{f}_n=\mathbf{f}(\mathbf{n})=-\mathbf{f}(-\mathbf{n})$ can be regarded as internal variable fields defined on Ω. On the other hand, their use makes the formulation too cumbersome for any practical application. The strategy applied in this paper for the reduction of the computational effort then uses approximate representations of such orientation fields.

First, as radially symmetric orientation function, the new damage variable $a_n=a(\mathbf{n})=\alpha_n^3$ is approximated by the first two nonzero terms of its expansion in a Fourier series of generalized spherical harmonics (Kanatani [10], He [11]), so allowing the form:

$$a(\mathbf{n})\approx a_I+\mathbf{n}\bullet\mathbf{A}'\mathbf{n}=\mathbf{n}\bullet\mathbf{A}\mathbf{n}\,, \tag{3}$$

where $\mathbf{A}$ is a symmetric second-order tensor ($\mathbf{A}^\mathrm{T}=\mathbf{A}$), denoted damage tensor; a_I and $\mathbf{A}'$ stand, respectively, for the hydrostatic (tr$\mathbf{A}=3a_I$) and deviatoric (tr$\mathbf{A}'$=0) components of $\mathbf{A}$. At the undamaged state $\mathbf{A}_0=\mathbf{0}$, whereas during the damage process $\mathbf{A}$ is positive definite. Note that a positive definite second-order tensor in particular reflects both the essential properties of load-induced anisotropy (namely, orthotropy) and irreversibility of damage.

Secondly, as regards contact tractions, a linear Coulomb condition and frictional sliding with no extension are assumed for each crack plane:

$$|\mathbf{f}_n|+\mu_n p_n\leq 0 \quad\text{together with}\quad p_n=-[-\sigma_n]\leq 0\,, \tag{4}$$

where μ_n is the friction coefficient and [•] denotes the McAuley operator. From eqn. (4) it follows that: (i) the normal tractions $p_n=p(\mathbf{n})$ vanish when tensile stresses act, that is on the set of orientations $\Omega^+ = \{\mathbf{n} \in \Omega \mid \mathbf{n}\bullet\mathbf{T}\mathbf{n} \geq 0\}$, whereas it depends explicitly on the mean stress tensor **T** on the set of compressive planes $\Omega^- = \Omega - \Omega^+$; (ii) the tangential tractions $\mathbf{f}_n$=**f**(**n**) equal the zero vector on planes with $\mathbf{n} \in \Omega^+$, while they turn out to be unknown on planes with $\mathbf{n} \in \Omega^-$. By analogy with shearing stresses, the field **f**(**n**) over Ω^- is approximated as follows:

$$\mathbf{f}(\mathbf{n}) \approx (\mathbf{I} - \mathbf{n} \otimes \mathbf{n})\mathbf{F}\mathbf{n} = (\mathbf{I} - \mathbf{n} \otimes \mathbf{n})\mathbf{F}'\mathbf{n}, \tag{5}$$

where **F**' is the only significant deviatoric component (tr**F**'=0) of the symmetric second-order tensor $\mathbf{F}=\mathbf{F}^{\mathrm{T}}$, denoted internal friction tensor, that =**0** if $\Omega^- \equiv \emptyset$.

Under the above assumptions, a simplified model where damage and friction tensors **A** and **F**' replace fields *a*(**n**) and **f**(**n**) as new internal variables is developed. The constitutive equation (1) can be shown to assume finally the form:

$$\mathbf{E} = \mathbb{K}\mathbf{T} + c_n \mathbb{H}_n^+[\mathbf{A}]\mathbf{T} + c_t \mathbb{H}_t^+[\mathbf{A}]\mathbf{T} + c_t \mathbb{H}_t^-[\mathbf{A}]\left(\mathbf{T} - \mathbf{F}\right), \tag{6}$$

where the matrix elastic moduli and constants c_n, c_t play the role of model parameters, whereas $\mathbb{H}_n^+$ and $\mathbb{H}_t^\pm$ are fourth-order crack opening and open/closed crack sliding compliance tensors defined by eqns (A.1-2) together with eqns (A.4-5). As suggested by the constitutive equation (6), postulating the following Gibbs free-energy function:

$$\Psi = \frac{1}{2}\left\{\mathbf{T}\bullet\mathbb{K}\mathbf{T} + c_n \mathbf{T}\bullet\mathbb{H}_n^+[\mathbf{A}]\mathbf{T} + c_t \mathbf{T}\bullet\mathbb{H}_t^+[\mathbf{A}]\mathbf{T} + c_t (\mathbf{T} - \mathbf{F})\bullet\mathbb{H}_t^-[\mathbf{A}](\mathbf{T} - \mathbf{F})\right\}, \tag{7}$$

permits one to associate **F**' and **A** with their thermodynamically conjugate variables, respectively: the closed crack sliding strain tensor $\mathbf{E}_t^-$ and the symmetric, positive definite second-order tensor **Y**, denoted damage energy release rate tensor:

$$\mathbf{Y} = \mathbf{Y}^+ + \mathbf{Y}^- = \frac{1}{2}\left\{c_n \mathbb{H}_n^+[\mathbf{T}]\mathbf{T} + c_t \mathbb{L}_t^+[\mathbf{T}']^{\mathrm{T}}\mathbf{T}'\right\} + \frac{1}{2}\left\{c_t \mathbb{L}_t^-[\mathbf{T}' - \mathbf{F}']^{\mathrm{T}}(\mathbf{T}' - \mathbf{F}')\right\}, \tag{8}$$

$\mathbb{L}_t^\pm$ being the fourth-order tensors given by eqn (A.3) together with eqns (A.4-5).

3 INTERNAL FRICTION AND DAMAGE EVOLUTION

In the framework of thermodynamics with internal variables, the proposed constitutive equation must be complemented by limit conditions, governing potential activation of dissipation mechanisms, and equations of evolution for the associated internal variables, describing the way irreversible processes evolve. According to the model proposed, the two dissipative micromechanisms of frictional contact between opposite faces of existing cracks and microcrack evolution, dominating the brittle constitutive response, are considered and described in terms of overall (tensor-valued) variables. So must be defined in a overall form the associated limit conditions and kinetic laws of activation and evolution of both irreversible processes of deformation.

As an example, generalizing the first equation of (4), a global sliding criterion is defined in terms of mean values of the contact tractions in the form:

$$\Phi_s = \sqrt{<|\mathbf{f}|^2>} + \mu < p > = \sqrt{\mathbf{F}'\bullet\mathbb{H}_t^-[\mathbf{I}]\mathbf{F}'} + \mu \mathbf{N}_2^- \bullet \mathbf{T} \leq 0, \tag{9}$$

where $\mathbf{N}_2^-$ is an orientation tensor given by eqn (A.6), <•> denotes average over Ω and μ represents an overall friction coefficient. Provided that the limit state $\Phi_s = 0$ is attained, the way sliding occurs is described by assuming the following evolution equation for the mean sliding strain due to closed cracks, thermodynamically associated with **F**':

$$\dot{\mathbf{E}}_t^- = \mathbf{V}_s \dot{\lambda}, \tag{10}$$

having defined the symmetric, traceless second-order tensor $\mathbf{V}_s$ as:

$$\mathbf{V}_s = \frac{\partial \Phi_s}{\partial \mathbf{F}'} = \frac{\mathbb{H}_t^-[\mathbf{I}]\mathbf{F}'}{\sqrt{\mathbf{F}'\bullet\mathbb{H}_t^-[\mathbf{I}]\mathbf{F}'}}, \tag{11}$$

whereas the unknown rate $\dot{\lambda} \geq 0$ can be obtained as solution of a Linear Complementarity Problem (LCP): $\dot{\Phi}_s \leq 0$, $\dot{\lambda} \geq 0$, $\dot{\Phi}_s \dot{\lambda} = 0$. It is worthwhile to note that the sliding condition is not active when tensile stresses act on all the crack planes.

Second, generalizing the Griffith criterion, the condition for damage evolution is defined as a global energy balance between the energy release rate **Y** and the material fracture toughness $R(\mathbf{A})$ in the following form:

$$\Phi_d = \sqrt{3}\left|\mathbf{Y}\right| - R(|\mathbf{A}|) = \sqrt{3}\left|\mathbf{Y}\right| - R(D) \leq 0 \quad \text{with} \quad D=1/3|\mathbf{A}|^2. \tag{12}$$

Other choices are certainly possible, but introduce a higher level of complexity to the formulation. A proper choice of the toughness function, which requires in general a phenomenological approach to determine, permits one to model the progressive stiffness degradation exhibited by brittle materials up to the limit strength and in the subsequent softening phase. Namely, R must first increase with increasing damage ($R'>0$) until a maximum R_c is attained at a critical damage D_c, then decrease ($R'<0$) until vanishing. Provided that the limit state $\Phi_d=0$ is attained, the following equation of damage evolution is assumed to describe the way microcracks propagate:

$$\dot{\mathbf{A}} = \mathbf{V}_d \dot{d}\,, \tag{13}$$

having defined the symmetric, positive definite (because of the irreversibility condition of damage) second-order tensor $\mathbf{V}_d$ as:

$$\mathbf{V}_d = \frac{\partial \Phi_d}{\partial \mathbf{Y}} = \sqrt{3}\frac{\mathbf{Y}}{|\mathbf{Y}|}. \tag{14}$$

The unknown rate $\dot{d} \geq 0$ is then obtained as solution of a LCP: $\dot{\Phi}_d \leq 0$, $\dot{d} \geq 0$, $\dot{\Phi}_d \dot{d} = 0$.

4 MODEL RESPONSE TO PROPORTIONAL LOADING

Because of nonlinearity and stress-path-dependence of the constitutive response, the model should be necessarily formulated in incremental form to predict the constitutive response to arbitrary stress applied to a brittle material at an arbitrary current state and requires, in general, numerical analyses to perform. Proportional loading is an exception and explicit solutions are possible as well.

In the case of proportional loading, the stress tensor at an arbitrary state can be expressed as $\mathbf{T} = \mathbf{T}_0\delta$, constant tensor $\mathbf{T}_0$ describing the load direction and loading multiplier $\delta \geq 0$ being positive. Two considerations follow. Firstly, the sets of tensile/compressive orientations are effectively governed by $\mathbf{T}_0$; thus, $\mathbf{T}_0$ being constant, so orientation tensors also, the status of the crack planes does not change during the loading program. Secondly, under the assumption of initially undamaged material, both internal variables can be shown to vary proportionally according to: $\mathbf{A} = \mathbf{A}_0 d$ and $\mathbf{F}' = \mathbf{F}_0'\delta$ with $\mathbf{A}_0$, $\mathbf{F}_0'$ constant tensors and $d\geq 0$. Despite stress-path-dependence of the constitutive response, from eqn (6) the model can be written in the final form:

$$\mathbf{E} = \delta\left\{\mathbb{K}\mathbf{T}_0 + d\left[c_n\mathbb{H}_n^+[\mathbf{A}_0]\mathbf{T}_0 + c_t\mathbb{H}_t^+[\mathbf{A}_0]\mathbf{T}_0' + c_t\mathbb{H}_t^-[\mathbf{A}_0](\mathbf{T}_0' - \mathbf{F}_0')\right]\right\}, \tag{15}$$

where damage parameter d must be regarded as function of δ obtained by solving the nonlinear limit damage state equation $\Phi_d=0$.

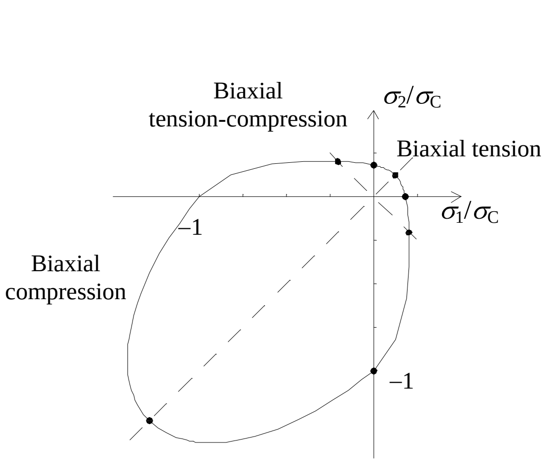

Figure 1. Biaxial limit domain.

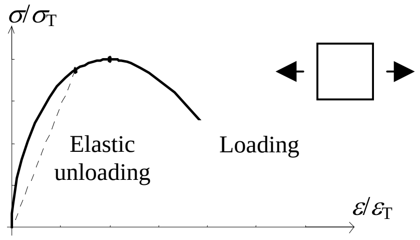

Figure 2. Model response to uniaxial tension.

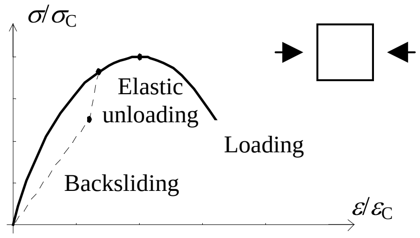

Figure 3. Model response to uniaxial compression.

On the basis of these results, biaxial and triaxial failure envelopes, together with some characteristic stress-strain curves, have been obtained and used for both identification and validation of the model. As an example, figures 1, 2 and 3 show the biaxial limit strength domain and some related stress–strain curves, computed for some values of model parameters. Note that: (i) nonlinearity of the overall responses reflects damage evolution, which in the specific case of compression tests is accompanied by frictional sliding; (ii) if unloading is initiated, since damage evolution is locked, a linearly elastic tensile response is observed, whereas in compression the bilinear unloading branch reflects the occurence of backsliding. These results in general confirm the previous Brencich and Gambarotta [8] simulations but the use here of a second-order tensor rather than a single scalar variable permits a more convenient description for the load-induced anisotropy (namely, orthotropy) of damage.

It is straightforward that, because of the assumption of statistically uniformly distributed damage as an ensemble of non-interacting microcracks, the model is inherently incapable of predicting the onset of macroscopic failure, occurring as damage localization into bands and coalescence of microdefects into a macrocrack. Practical application of the model is then restricted to the hardening phase of the mechanical response of brittle and quasi-brittle materials. Furthermore, due to the self-similar crack growth, the characteristic feature of dilatancy cannot be captured. On the other hand, with all of its limitations, in this model effects of anisotropic damage and frictional sliding for prescribed (even complex) loading are combined in a relatively simple formulation which takes the physics of the problem into account.

5 REFERENCES

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6 APPENDIX

For the sake of brevity, details of the approach are omitted and only the final explicit expressions for the inelastic compliance tensors are given [12]:

$$\mathbb{H}_n^+[\mathbf{A}]\mathbf{T} = \mathbb{H}_n^+[\mathbf{T}]\mathbf{A} = (\mathbb{N}_6^+\mathbf{A})\mathbf{T} = (\mathbb{N}_6^+\mathbf{T})\mathbf{A} \ , \tag{A.1}$$

$$\mathbb{H}_t^\pm[\mathbf{A}]\mathbf{T} = \mathbb{H}_t^\pm[\mathbf{A}]\mathbf{T}' = \left[\frac{1}{2}\left(\mathbf{I}\boxdot\mathbb{N}_4^\pm\mathbf{A} + \mathbb{N}_4^\pm\mathbf{A}\boxdot\mathbf{I}\right) - \mathbb{N}_6^\pm\mathbf{A}\right]\mathbf{T}' \ , \tag{A.2}$$

$$\mathbb{L}_t^\pm[\mathbf{T}]^{\mathrm{T}} = \mathbb{L}_t^\pm[\mathbf{T}']^{\mathrm{T}} = \left[\frac{1}{2}\mathbb{N}_4^\pm\left(\mathbf{I}\boxdot\mathbf{T}' + \mathbf{T}'\boxdot\mathbf{I}\right) - \mathbb{N}_6^\pm\mathbf{T}'\right] \quad \text{such that} \quad \mathbb{L}_t^\pm[\mathbf{T}]^{\mathrm{T}}\mathbf{T} = \mathbb{L}_t^\pm[\mathbf{T}]^{\mathrm{T}}\mathbf{T}' \ , \tag{A.3}$$

having introduced the notation $\left(\mathbf{B}\boxdot\mathbf{C}\right)\mathbf{X} = \mathbf{B}(\mathbf{X}+\mathbf{X}^{\mathrm{T}})\mathbf{C}^{\mathrm{T}}/2$ and being $\mathbb{N}_6^\pm$, $\mathbb{N}_4^\pm$ and $\mathbf{N}_2^\pm$ the following sixth-, fourth- and two-order orientation tensors:

$$\mathbb{N}_6^\pm = \frac{1}{2\pi}\int_{\Omega^\pm}(\mathbf{n}\otimes\mathbf{n}\otimes\mathbf{n}\otimes\mathbf{n}\otimes\mathbf{n}\otimes\mathbf{n})\mathrm{d}\Omega \, , \tag{A.4}$$

$$\mathbb{N}_4^\pm = \mathbb{N}_6^\pm\mathbf{I} = \frac{1}{2\pi}\int_{\Omega^\pm}(\mathbf{n}\otimes\mathbf{n}\otimes\mathbf{n}\otimes\mathbf{n})\mathrm{d}\Omega \, , \tag{A.5}$$

$$\mathbf{N}_2^\pm = \mathbb{N}_4^\pm\mathbf{I} = \frac{1}{2\pi}\int_{\Omega^\pm}(\mathbf{n}\otimes\mathbf{n})\mathrm{d}\Omega \, . \tag{A.6}$$