A WAVELET BASED VORONOI CELL FEM FOR CRACK EVOLUTION IN COMPOSITE MATERIALS

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ABSTRACT

This paper describes the development of the Voronoi cell finite element model (VCFEM) enriched by multi-resolution wavelet functions for modeling microstructures with cracks, particularly in composite materials. Additionally, h-p adaptation for displacements is employed to reduce traction discontinuity along the element boundaries and stress oscillation around crack tips. Besides wavelet basis functions, a set of particular branch functions based on level set methods is introduced to describe the discontinuity across the crack. A special numerical method is invoked for stress-based elements to assess the J-integral. This is a way to obtain the stress intensity factors (K_I and K_{II}). Comparison of stress intensity factors of several classic fraction mechanical problems between VCFEM simulations and theoretical predictions demonstrates the feasibility and accuracy of the wavelet-based VCFEM. Furthermore, a cohesive zone model is added for representing crack propagation. The magnitude of the growth per step is determined by adjusting the external loading according to the arc length algorithm.

1 INTRODUCTION

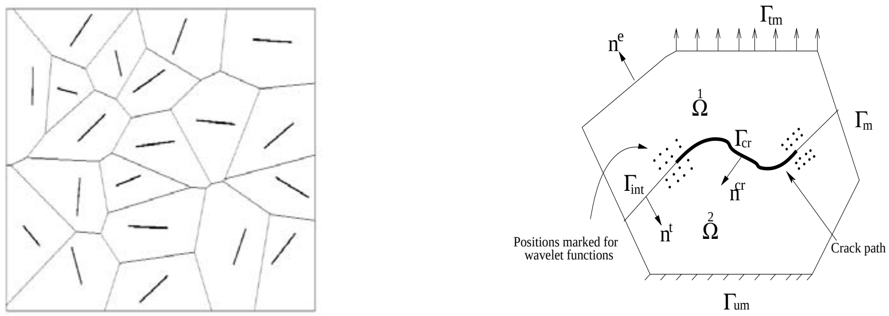
Over the past a few decades, more and more emphasis is placed on material cracking. After Westergaard[1] provided a semi-inverse method to solve a certain class of plane elasticity problems, the finite element method has also been used in determining stress intensity factor and stress distribution by numerous authors Galagher[2]. It is recognized that the conventional finite element method converges slowly in applications about crack problem. Since the stress singularity is not included in element formulation, a very fine mesh is necessary near the crack tip. In order to improve the computational efficiency, singular elements, such as quarter-point element were developed to describe the singular stress field with numerical methods. In contrast to these finite elements based on the displacement interpolation, a super-element containing a crack developed in Tong, Pian and Lasry[3] was based on the hybrid finite element method and classical elastic theory to model crack media. At present, another super-element appeared in the work of Belytschko and coworkers [Belytschko and Black[4], Belytschko, Organ and Gerlach[5]]. The meshless approach and extended finite element methods have been developed to model arbitrary discontinuities without refine mesh. In fact, the linear elastic fracture mechanics (LEFM) is only suitable when the size of the fracture process zone at the crack tip is small compared with the size of the specimen. Also since an infinite stress magnitude never appears in a true specimen, an equivalent concentrated stress field is more reasonable. A type of Voronoi cell finite element (VCFE) enriched by multi-resolution wavelet functions is introduced to replace the singular crack tip in LFEM with a region with stress concentrations. Voronoi cell finite element model (VCFEM) is established by Moorthy and Ghosh[6] as an effective tool for modeling non-uniform microstructures efficiently and accurately. And it was further improved by adding cohesive zone models for describing the interfacial debonding in (Ghosh, Ling[7]; Li, Ghosh[8]). The Voronoi cell formulation is improved in this paper to incorporate the stress distribution around a crack and crack growth by introducing wavelet functions and cohesive zone models. Due to its multi-resolution property, wavelet leads to efficient methods for numerical solution of differential equations, such as directly coupling with Galerkin's method (Qian and Weiss[9]).

2 LINEAR ELASTIC FRACTURE WITH THE VORONOI CELL FEM

The VCFEM for a heterogeneous domain with a dispersion of inclusions or voids and cracks, implements a mesh of Voronoi polytopes. Each element in VCFEM consists of the heterogeneity and its neighborhood region of matrix. VCFEM is successfully improved in Moorthy and Ghosh[6] and Ghosh, Ling[7]. Consider a typical representative volume element Ω consisting of N cracks (Fig. 1(a)), each contained in a Voronoi cell element Ω_e (see Fig 1(b)). A complementary energy functional in incremental form for one element may be given in terms of increments of stress and boundary displacement fields as:

$$
\begin{aligned}
\Pi_e = {} & \frac{1}{2}\int_{\overset{1}{\Omega}_e} (\overset{1}{\sigma}_{ij}+\Delta\overset{1}{\sigma}_{ij}) S_{ijkl} (\overset{1}{\sigma}_{ij}+\Delta\overset{1}{\sigma}_{ij})\, d\overset{1}{\Omega} + \frac{1}{2}\int_{\overset{2}{\Omega}_e} (\overset{2}{\sigma}_{ij}+\Delta\overset{2}{\sigma}_{ij}) S_{ijkl} (\overset{2}{\sigma}_{ij}+\Delta\overset{2}{\sigma}_{ij})\, d\overset{2}{\Omega} - \int_{\partial\overset{1}{\Omega}_e} (\overset{1}{\sigma}_{ij}+\Delta\overset{1}{\sigma}_{ij})\, \overset{1}{n}{}_j^e (\overset{1}{u}{}_i^e+\Delta\overset{1}{u}{}_i^e)\, d\partial\overset{1}{\Omega} \\
& -\int_{\partial\overset{2}{\Omega}_e} (\overset{2}{\sigma}_{ij}+\Delta\overset{2}{\sigma}_{ij})\, \overset{2}{n}{}_j^e (\overset{2}{u}{}_i^e+\Delta\overset{2}{u}{}_i^e)\, d\partial\overset{2}{\Omega} - \int_{\Gamma_{cr}} (\overset{1}{\sigma}_{ij}+\Delta\overset{1}{\sigma}_{ij})\, \overset{1}{n}{}_j^{cr} (\overset{1}{u}{}_i^e+\Delta\overset{1}{u}{}_i^e)\, d\Gamma_{cr} + \int_{\Gamma_{cr}} (\overset{2}{\sigma}_{ij}+\Delta\overset{2}{\sigma}_{ij})\, \overset{2}{n}{}_j^{cr} (\overset{2}{u}{}_i^e+\Delta\overset{2}{u}{}_i^e)\, d\Gamma_{cr} \\
& -\int_{\Gamma_{\mathrm{int}}} (\overset{1}{\sigma}_{ij}+\Delta\overset{1}{\sigma}_{ij}-\overset{2}{\sigma}_{ij}-\Delta\overset{2}{\sigma}_{ij})\, n_j^t (u_j^t+\Delta u_j^t)\, d\Gamma_{\mathrm{int}} + \int_{\overset{1}{\Gamma}_{tm}} \overset{1}{\bar{T}}_i (\overset{1}{u}{}_i^e+\Delta\overset{1}{u}{}_i^e)\, d\overset{1}{\Gamma}_{tm} + \int_{\overset{2}{\Gamma}_{tm}} \overset{2}{\bar{T}}_i (\overset{2}{u}{}_i^e+\Delta\overset{2}{u}{}_i^e)\, d\overset{2}{\Gamma}_{tm} \\
& -\int_{\Gamma_{cr}} \int_0^{\overset{2}{\mathbf{u}}+\Delta\overset{2}{\mathbf{u}}-\overset{1}{\mathbf{u}}-\Delta\overset{1}{\mathbf{u}}} T_i^{coh}\, d(\overset{2}{u}_i+\Delta\overset{2}{u}_i-\overset{1}{u}_i-\Delta\overset{1}{u}_i)\, d\Gamma_{cr} \qquad (1)
\end{aligned}
$$

In two-dimensional analysis, Airy's stress function $\Phi(x,y)$ is a convenient tool for deriving equilibrated stress fields. In view of the existent of cracks, the stress functions are decomposed into (a) a purely polynomial function Φ^{poly}, (b) a special branch function Φ^{brch}, and (c) a multi-resolution wavelet function Φ^{wlt} ($\Phi=\Phi^{poly}+\Phi^{brch}+\Phi^{wlt}$). Firstly, the pure polynomial function Φ^{poly} is written in the complete polynomial form, which is continuous in each element. Secondly, the set of particular branch functions Φ^{brch} based on the level set method is introduced to describe the discontinuity across the crack. Since the crack path is represented as the zero level set of a time-dependent, implicit function, even specimens with curvilinear cracks can be easily modeled. A general overview of the theory, numerical approximation, and range of applications may be found in Sethian[10].



(a) (b)

Fig. 1 (a) A mesh of Voronoi cell elements generated by tessellation of the heterogeneous miscrostructural domain. (b) A typical Voronoi cell element enriched by wavelet functions.

In Belytschko[11], level sets were used to update the position of the discontinuities for crack growth, and the surfaces of discontinuity was described by signed distance function. In this paper, since VCFEM requires that the second order derivatives of branch functions have to be continuous in each element except on crack paths, we approach the crack path with the cubic spline interpolation algorithm. Based on the level set method and the cubic spline interpolation, the branch function is continuous in each element except on crack paths. Besides the pure polynomial function and the branch function, wavelet basis functions Φ^{wlt} are adaptively enriched to accurately capture crack-tip stress concentrations, necessary for representing microstructural damage in composites. Unlike trigonometric approximation, approximation with wavelet bases does not rely on cancellation. When an abrupt change, such as a shock wave, occurs in a function, only local coefficients in a wavelet approximation will be affected. The localization property of wavelet functions makes the wavelet technique a powerful tool in problems with a high gradient, such as stress concentration, even singularity. During past several decades, people discussed the use of various wavelet bases for numerical solutions of ODEs and PDEs, such as Daubechies' orthonormal bases and Chui-Wang's semi-orthogonal B-spline wavelets. Comparisons of wavelets methods with traditional methods show that wavelets are advantageous for problems with multi-level features when wavelet basis functions are embedded in the method of weighted residuals. Daubechies wavelets have compact support and are orthogonal, but they do not have an explicit or analytic mathematical expression. This makes it difficult to obtain their derivatives. So Daubechies wavelets are not suitable as interpolation bases of stress functions. Chui-Wang's

B-spline wavelets have compact support and analytical expressions and are semi-orthogonal, but their derivatives no longer keep the semi-orthogonal property. The loss of orthogonality implies that the fast algorithm becomes infeasible and the accuracy can't be guaranteed. Except for the previous two kinds of wavelet bases, another choice is the family of Gaussian function, which is expressed as: $G(x) = e^{-\left(\frac{x-b}{a}\right)^2/2}$. The Gaussian function owns a simple analytical expression, and the derivatives of Gaussian function are popular wavelets bases (Brasseur, Wang[12]). The two parameters a and b translate and dilate the basic function. The translation parameter b provides the ability to move the basic wavelet prototype around the entire domain, and the dilation parameter a controls the ability to catch the details. The smaller the dilation parameter, the smaller the critical space/time step becomes and the more accurate localized details are picked up.

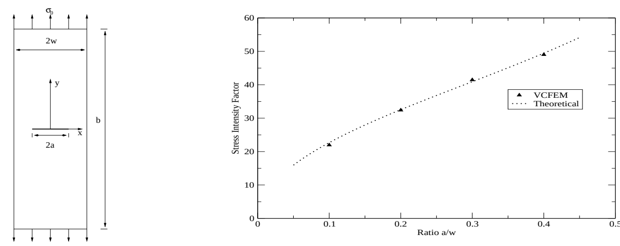
In this paper, a cohesive zone model is added for representing crack propagation. This produces a concentrated stress field around the crack tip instead of a singular one. Cohesive zone models are effective in depicting material failure as a separation process across an extended crack tip. The tractions across the crack reach a maximum, subsequently decrease and eventually vanish with increasing crack separation. Bilinear cohesive model is coupled into VCFEM to describe the crack growth. During cohesive crack growth, snap-back may occur. The processing controlled monotonically by deformation will ignore the snap-back part and the solution curve will show a discontinuity with a negative jump. The method revealing the snap-back part is to decrease both load and deformation while the crack grows and opens. In general, Newton-Raphson solver cannot catch the snap-back branch, since the loading processing is monotonically controlled by external deformation or load conditions. The arc-length solver has been proposed in Crisfield[13] as a method of overcoming this shortcoming by introducing an arc length as a replacement to the incremental load as the incremental parameter.

Unlike the super-element model, post process for calculating stress intensity factors is necessary in VCFEM. Yau, Wang and Corten[14] provided a method to extract the stress intensity factors K_I and K_{II} from a mixed-mode crack by using J-integral. The least square method is used to approach the contour integral by evaluating the gradient of a displacement field.

3 NUMERICAL EXAMPLES

3.1 Center cracked panel under remote tension loading for various crack length

A center cracked panel under remote tension load for various crack length is calculated. As shown in Fig. 2(a), a center cracked plate with a crack length 2a=0.8m is under the loading simple tension, where Young's modulus E=1MPa, Poisson ratio ν=0.3, width 2w=2m, length b=6m, and far field tension load σ_0=40Pa. In order to compare with the theoretical prediction (Tada[15]), the numerical results of K_I based on VCFEM for various cases of a/w are marked in Fig. 2(b). As illustrated in Fig. 2(b), VCFEM predicts ac ... ick length.

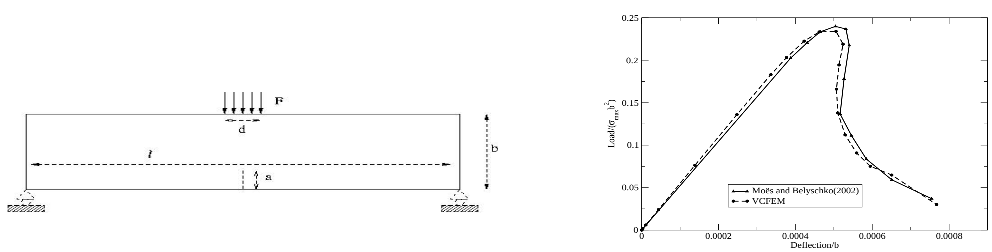


(a) (b)

Figure 2 (a) A center cracked panel under remote tension loading, (b) stress intensity factors for various cases of a/w

3.2 A three-point bending beam

As shown in Fig. 3(a), a three-point bending beam is considered. The propagation of a cohesive crack in such a beam has been studied in Moës and Belytschko[16] using node release technique and XFEM, respectively. The geometrical parameters for the specimen in Fig. 9 are $b = 0.15m$, $l = 4b$, $t = b$, $a = 0$, and $d = 0.01m$ (t is the specimen thickness). The material properties are $E = 36,500\text{MPa}$, $\nu = 0.1$, According to Moës and Belytschko[16], the fracture energy is $G_F = 50\text{Nm}^{-1}$. The corresponding parameter in this paper is $\delta_e = 3.134796 \times 10^{-5}$ m. The load-deflection points obtained are shown in Fig. 3(b). We observe that the VCFEM results are consistent with the referenc



(a) (b)

Fig. 9. (a) A three-point bending beam. (b) Normalized load-deflection curves.

4 CONCLUDING REMARKS

A special crack-tip element is developed for the analysis of 2-D crack problems. The key feature of this method is the use of wavelet functions, which are a kind of multi-resolution functions. The wavelet functions are placed around the crack tip, so that the crack is not treated by the pure polynomial basis. As shown in this paper, the results express excellent accuracy for a range of plane problems. After incorporating with cohesive zone models and the arc-length method, the improved VCFEM described the entire damage process of a three-point bending beam, even with a sharp snap-back.

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