A GRADIENT DAMAGE MODEL FOR DUCTILE FRACTURE

E. Lorentz [1] and V. Cano [2]
[1] Laboratoire de Mécanique des Structures Industrielles Durables, EDF / CNRS, Clamart, 92140 FRANCE
[2] Département Analyses Mécaniques et Acoustique, EDF R&D, Clamart, 92140 FRANCE

ABSTRACT
The void growth responsible for ductile fracture is described by the Rousselier model. It has been revisited so that it reduces to a standard plastic law where the void effect is taken into account through the total volume change. Thanks to a specific finite strain description, the constitutive law is expressed as a minimisation problem. Therefore, it can fully benefit from a variational formulation dedicated to gradient constitutive laws. The introduction of the gradient of the cumulated plasticity field into the constitutive equations efficiently controls the localisation process, as demonstrated by the numerical simulation of a notched specimen.

1 INTRODUCTION

This work aims at predicting the ductile fracture of real-life metallic structures, from the inception of damaged zones to their ultimate propagation. Two points are explored in this study :

- the ductile fracture law which models plastic void growth,
- the nonlocal formulation to control the localisation phenomenon,

The nonlocal formulation is based on the introduction of the gradient of the cumulated plasticity field inside the constitutive law. It relies on a principle of energy minimisation which gives rise to interesting properties ranging from the physics up to numerical solutions, see Lorentz and Andrieux [1]. However, it requires a strong assumption : the constitutive law should belong to the framework of generalised standard materials. Therefore, we choose to begin with the Rousselier model [2] and to revisit it so that it is reduced to a classical plastic law while preserving its main features. A specific finite strain formulation then allows to express the constitutive law as a minimum principle.

A numerical simulation of a notched specimen finally demonstrates the potency of the method.

2 THE ROUSSELIER MODEL REVISITED

2.1 The original Rousselier model

On the basis of microstructural assumptions, Rousselier [2] proposed a constitutive law to describe ductile fracture. Stated in a finite strain formalism, it combines a plastic law with isotropic hardening and a damage evolution law related to the void growth.

More precisely, the material state is described by the total deformation gradient, the plastic strain, the hardening variable p and the damage variable β ranging from 0 (initial state with porosity f_0) to $+\infty$ (ultimate stage with porosity equal to 1). The evolution of the internal variables is governed by the following threshold :

$$\mathrm{g_R}\left(\boldsymbol{\tau}, A, B\right) = \tau_{eq} + B\,D\exp\left(\frac{\mathrm{tr}\,\boldsymbol{\tau}}{3\sigma_1}\right) + A - \sigma^y \quad \leq \quad 0 \tag{1}$$

$\boldsymbol{\tau}$ denotes Kirchhoff stress tensor, σ^y is the yield stress, σ_1 and D are material parameters in relation with volumetric plasticity and damage. Isotropic hardening is introduced through the driving force A associated to the hardening variable p ; it is a function of p only :

$$A = -\mathrm{R}\left(p\right) \tag{2}$$

where R is the usual hardening function. Damage is introduced through the driving force B associated to the damage variable β ; it is a function of β only :

$$B = \frac{\sigma_1 f_0 \exp\beta}{1 - f_0 + f_0 \exp\beta} \tag{3}$$

The eulerian plastic rate $\mathbf{D^p}$, the cumulated plastic strain rate $\dot{p}$ and the damage rate $\dot{\beta}$ are given by a normality rule with respect to the yield surface complemented by the consistency condition :

$$\mathbf{D^p} = \lambda \frac{\partial g_R}{\partial \boldsymbol{\tau}} \qquad \dot{p} = \lambda \frac{\partial g_R}{\partial A} \qquad \dot{\beta} = \lambda \frac{\partial g_R}{\partial B} \tag{4}$$

$$\lambda \geq 0 \qquad g_R\left(\boldsymbol{\tau}, A, B\right) \leq 0 \qquad \lambda\, g_R\left(\boldsymbol{\tau}, A, B\right) = 0 \tag{5}$$

Finally, the model is completed by a stress – strain relation which is usually provided in a rate form (with Jaumann's rate, for instance) :

$$\overset{\circ}{\boldsymbol{\tau}} = \mathbf{E} : \left(\mathbf{D} - \mathbf{D^p}\right) \tag{6}$$

where **D** is the eulerian strain rate and **E** is Hooke's tensor.

2.2 Reducing the Rousselier model to a plastic constitutive law

To apply the nonlocal variational formulation described in Lorentz and Andrieux [1], the Rousselier model has to be recast in the framework of generalised standard materials. It requires in particular that the yield threshold be convex, a property which is not fulfilled by eqn (1). Therefore, the significance of the damage variable should be revisited to reduce the model to a plastic law.

Ductile constitutive laws may be expressed in terms of the porosity f instead of a new damage variable. The evolution of the porosity then derived from some microscopic assumptions. If coalescence is neglected, a relation widely used reads :

$$\dot{f} = \left(1 - f\right) \operatorname{tr} \mathbf{D^p} \qquad \text{with} \qquad f\left(t = 0\right) = f_0 \tag{7}$$

The flow rule eqn (4) provides a relation between the damage rate $\dot{\beta}$ and the volumetric plastic strain rate :

$$\begin{cases} \operatorname{tr} \mathbf{D^p} = \lambda \dfrac{f_0 \exp\beta}{1 - f_0 + f_0 \exp\beta} D \exp\left(\dfrac{\operatorname{tr} \boldsymbol{\tau}}{3\sigma_1}\right) \\ \dot{\beta} = \lambda D \exp\left(\dfrac{\operatorname{tr} \boldsymbol{\tau}}{3\sigma_1}\right) \end{cases} \Rightarrow \quad \operatorname{tr} \mathbf{D^p} = \frac{f_0 \exp\beta}{1 - f_0 + f_0 \exp\beta} \dot{\beta} \tag{8}$$

Combining eqns (7) and (8) leads to the following simple expression for the driving force B :

$$B = \frac{\sigma_1 f_0 \exp\beta}{1 - f_0 + f_0 \exp\beta} = \sigma_1 f \tag{9}$$

In that way, the role of the damage variable may be replaced by the porosity. The yield threshold then reads :

$$g_f\left(\boldsymbol{\tau}, A, f\right) = \tau_{eq} + \sigma_1 f D \exp\left(\frac{\operatorname{tr} \boldsymbol{\tau}}{3\sigma_1}\right) + A - \sigma^y \quad \leq \quad 0 \tag{10}$$

Of course, relying on the threshold eqn (10) along with the porosity evolution law eqn (7) remains strictly equivalent to the model described in the previous section. Now, we go one step further and eliminate the porosity. Since the model is dedicated to metal behaviour for which the elastic strain is negligible compared to the plastic strain, the porosity evolution may also be stated in terms of the strain rate $\mathbf{D}$ instead of the plastic strain rate $\mathbf{D^p}$:

$$\dot{f} = \left(1 - f\right) \operatorname{tr} \mathbf{D} \tag{11}$$

Eqn (11) admits an integrated form leading to the following explicit expression :

$$f = 1 - \frac{1 - f_0}{J} \tag{12}$$

where J denotes the volumetric strain, equal to the Jacobian of the deformation. Our initial aim is achieved : the plastic threshold has become a function of the stress, the hardening variable and the volumetric strain J which retains all the void growth damage effect.

2.3 Finite strain formulation

The finite strain formulation applied in the initial Rousselier model is based on a normality rule expressed in terms of the eulerian plastic rate $\mathbf{D^p}$. The drawback of such a formulation is the lack of incremental objectivity. Moreover, the constitutive law cannot be formulated as the minimum of an energy, on the contrary of the corresponding infinitesimal strain model. Therefore, Lorentz and Cano [3] introduced an extension of Simo and Miehe finite strain theory [4]. It relies on three assumptions.

- A multiplicative split of the deformation gradient **F** in a plastic part $\mathbf{F^p}$ and an elastic part $\mathbf{F^e}$ through the introduction of a relaxed configuration (where the stress is zero) :
$$\mathbf{F} = \mathbf{F^e}\,\mathbf{F^p} \tag{13}$$
- A partition of the free energy in two terms : the stored energy Φ^{st} which depends on the hardening variable p and the elastic energy Φ^{el} that depends only on the elastic strain. Actually, the latter can be expressed without loss of generality as a function of an eulerian strain measure **e** since the elasticity is isotropic. The partition reads (with **E** Hooke's tensor) :
$$\Phi(\mathbf{e},p) = \Phi^{el}(\mathbf{e}) + \Phi^{st}(p) \quad \text{with} \quad \mathbf{e} = \frac{1}{2}\left(\mathbf{Id} - \mathbf{F^e}\,\mathbf{F^{eT}}\right) \tag{14}$$
$$\Phi^{el}(\mathbf{e}) = \frac{1}{2}\mathbf{e} : \mathbf{E} : \mathbf{e} \qquad \Phi^{st}(p) = \int_0^p \mathrm{R}(u)\,du \tag{15}$$
Thermodynamic consistency leads to the following plastic strain definition, driving force definitions and stress – strain relation :
$$\mathbf{G^p} \underset{def}{=} \left(\mathbf{F^{pT}}\,\mathbf{F^p}\right)^{-1} \tag{16}$$
$$\mathbf{s} \underset{def}{=} -\frac{\partial \Phi}{\partial \mathbf{e}} = -\mathbf{E} : \mathbf{e} \qquad A \underset{def}{=} -\frac{\partial \Phi}{\partial a} = -\mathrm{R}(p) \tag{17}$$
$$\boldsymbol{\tau} = \mathbf{s}\left(\mathbf{Id} - 2\mathbf{e}\right) \tag{18}$$
- The definition of the yield surface as a convex function of the driving forces which may also depend on the deformation gradient as a parameter :
$$\mathrm{g}(\mathbf{s}, A\,;J) = s_{eq} + \sigma_1 D\,f \exp\left(\frac{\mathrm{tr}\,\mathbf{s}}{3\sigma_1}\right) + A - \sigma^y \tag{19}$$
It should be noticed that Kirchhoff stress tensor $\boldsymbol{\tau}$ has been replaced by the driving force stress **s**. This is of small quantitative effect since they are close to each other, thanks to eqn (18) and to the fact that the elastic strain **e** remains small. The principle of maximal plastic dissipation with respect to the yield surface g leads to the evolution law and the consistency condition :
$$-\frac{1}{2}\mathbf{F}\,\dot{\mathbf{G}}^{\mathbf{p}}\,\mathbf{F}^{\mathrm{T}} = \dot{p}\frac{\partial \mathrm{g}}{\partial \mathbf{s}} \tag{20}$$
$$\dot{p} \geq 0 \quad \mathrm{g}(\mathbf{s}, A\,;J) \leq 0 \quad \dot{p}\,\mathrm{g}(\mathbf{s}, A\,;J) = 0 \tag{21}$$

2.4 Time-integration as a minimisation of an energy

The revisited Rousselier model as decribed by eqns. (16)-(21) appears as a classical plasticity model. Therefore, we choose to base the time-discretisation on an implicit backward Euler scheme which reads formally :
$$\mathrm{f}(\dot{q}, q, t) = 0 \quad \xrightarrow[\text{the time-step } \left[t^n\ t^{n+1}\right]]{\text{discretisation over}} \quad \mathrm{f}\left(\frac{\Delta q}{\Delta t}, q^{n+1}, t^{n+1}\right) = 0 \tag{22}$$

Thanks to some properties of the finite strain formulation, the solution $\left(\mathbf{e}^{n+1}, p^{n+1}\right)$ of the corresponding non linear equations for a given total strain **F** admits a variational characterisation :

$$\left(\mathbf{e}^{n+1}, p^{n+1}\right) = \arg\min_{\mathbf{e},\, p} \left[\Phi\left(\mathbf{e}, p\right) + \Psi\left(\mathbf{e} - \mathbf{e}^E, p - p^n \; ; J^{n+1}\right) \right] \tag{23}$$

where $\mathbf{e}^E$ stands for the elastic trial, that is the value that the elastic strain would have without plastic correction during the current time-step. The dissipation potential Ψ is defined through a Legendre transform of the yield function g :

$$\Psi\left(\mathbf{D^p}, \dot{p} \; ; J\right) \underset{def.}{=} \max_{\substack{\mathbf{s}, A \\ \mathrm{g}(\mathbf{s}, A \, ; \, J) \le 0}} \left(\mathbf{s} : \mathbf{D^p} + A\,\dot{p}\right) \tag{24}$$

The main objective of this part is achieved : the proposition of a ductile fracture constitutive law which can be characterised through the minimisation of an energy. This allows the application of the nonlocal variational formulation of next part.

3 NONLOCAL FORMULATION

3.1 Variational principle

To control the high spatial variations of the mechanical fields resulting from localisation, a gradient law is derived from the Rousselier local law. We choose to introduce the gradient of the hardening variable *p*, which proves sufficient to stabilise the localisation of all the mechanical fields, including the porosity. On the basis of Andrieux et al. [5], the gradient is introduced into the free energy through a quadratic term at the structural level, resulting in new definitions of a global free energy F and a global dissipation potential D which depend on the fields of state variables :

$$\mathrm{F}\left(\mathbf{e}, p\right) = \int_\Omega \left[\Phi\left(\mathbf{e}(x), p(x)\right) + \frac{1}{2} c\, \nabla p(x) \cdot \nabla p(x) \right] dx \tag{25}$$

$$\mathrm{D}\left(\mathbf{D^p}, \dot{p} \; ; J\right) = \int_\Omega \Psi\left(\mathbf{D^p}(x), \dot{p}(x) \; ; J(x)\right) dx \tag{26}$$

where Ω denotes the body domain in the initial configuration and *c* a new material parameter related to a characteristic length. It should be noted that the gradient of the hardening variable is expressed in the initial configuration. If it was expressed in the current configuration, a push-forward tensor rule should be applied and would transform the effective characteristic length. Actually, it would even involve induced anisotropy for the gradient terms.

The minimisation characterisation eqn (23) remains applicable at the structural scale, see Lorentz and Andrieux [1999]. It is now expressed in function spaces since the variables are fields and no more pointwise values. It results in the following minimisation problem for a given strain field **F** :

$$\left(\mathbf{e}^{n+1}, p^{n+1}\right) = \arg\min_{\mathbf{e},\, p} \left[\mathrm{F}\left(\mathbf{e}, p\right) + \mathrm{D}\left(\mathbf{e} - \mathbf{e}^E, p - p^n \; ; J^{n+1}\right) \right] \tag{27}$$

The minimisation completely characterises the numerical integration of the nonlocal constitutive law. It has to be complemented by the usual equilibrium equations to find the displacement at the end of the current time-step, as is done classically with a local constitutive law.

3.2 Pointwise interpretation

The minimum in eqn (27) admits a straightforward characterisation. First, bulk equations are derived, that can be expressed in terms of state equations, a flow equation and a consistency condition :

$$\mathbf{s}^{n+1} = -\,\mathbf{E} : \mathbf{e}^{n+1} \qquad \overline{A}^{n+1} = -\,\mathrm{R}\left(p^{n+1}\right) + c\,\nabla^2 p^{n+1} \tag{28}$$

$$\mathbf{e}^{n+1} - \mathbf{e}^E = \Delta p \frac{\partial \mathrm{g}}{\partial \mathbf{s}} \left(\mathbf{s}^{n+1}, \overline{A}^{n+1} \; ; J^{n+1}\right) \tag{29}$$

$$\Delta p \ge 0 \quad \mathrm{g}\left(\mathbf{s}^{n+1}, \overline{A}^{n+1} \; ; J^{n+1}\right) \le 0 \quad \Delta p\, \mathrm{g}\left(\mathbf{s}^{n+1}, \overline{A}^{n+1} \; ; J^{n+1}\right) = 0 \tag{30}$$

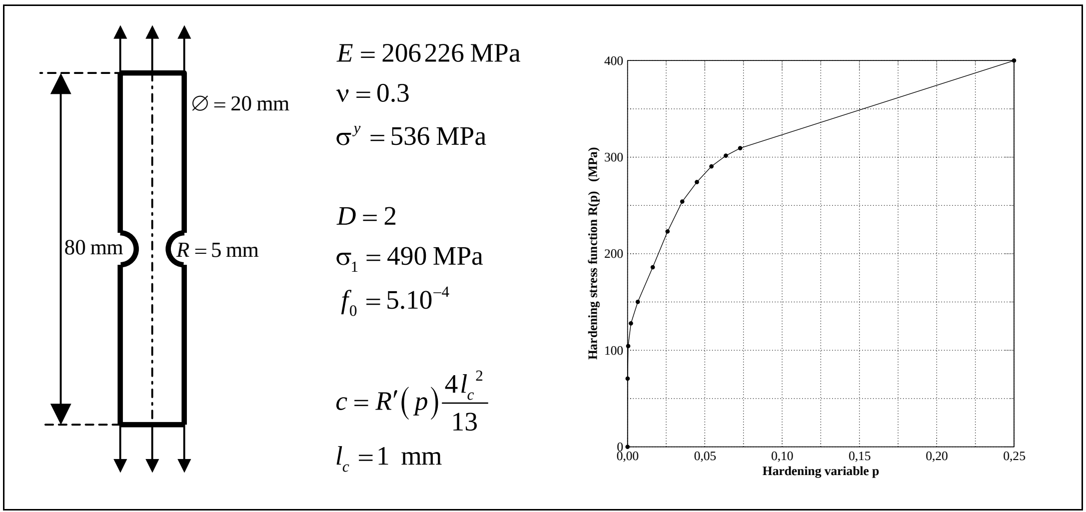

figure 1 - Geometry, loading and material parameters

The only nonlocal term is related to the introduction of the nonlocal driving force $\bar{A}$ with consequences into the differential inequation (30) : the Laplacian of the hardening variable appears in the yield threshold. Thus, the introduction of the quadratic gradient term into the global free energy eqn (26) leads to the gradient model family introduced by Muhlhaus and Aïfantis [6].

The minimum characterisation also provides natural boundary conditions and interface conditions. Namely, the gradient flux has to vanish along the domain boundary $\partial\Omega$:

$$\nabla p \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega \tag{31}$$

where **n** denotes the outer normal to the boundary. Moreover, two interface conditions appear, one related to the necessary regularity of the hardening variable field (so that the free energy be finite) and the other to the minimum characterisation. Across any interface *I* of normal $\mathbf{v}$, one has :

$$[\![p]\!] = 0 \qquad [\![\nabla p]\!] \cdot \mathbf{v} = 0 \tag{32}$$

It should be noted that the boundary and the interface conditions are stated on the hardening variable itself and not on its rate, which is a difference with many authors to whom a condition is missing to complete their set of equations.

3.3 Numerical application

To analyse the characteristics of such a gradient law, a numerical simulation is carried out. A notched specimen is submitted to tension, see figure 1 for the geometry, the loading and the material parameters. The computations benefit from the axial and the plane symmetries. They are led with two meshes, one with 0.25 mm element size in the localisation zone (coarse mesh), and the other with 0.1 mm element size. The minimisation eqn (27) is handled by the algorithm presented in Lorentz and Benallal [7], which has already proven its robustness in the context of gradient brittle damage simulations.

The numerical results are presented in figure 2 in terms of the porosity distribution during the propagation of a damaged zone. They are also compared to purely local computations. A good independence of the results with respect to the mesh refinement is observed, on the contrary of the local model. The localisation band spreads over several finite elements (Gauss point visualisation to avoid any artefact related to a graphical extrapolation) whose width is not set by the mesh refinement.

These results seem promising regarding the potencies of this gradient law. Other computations are at stage to evaluate the sensitivity to other numerical parameters such as the time discretisation, the convergence residual threshold, the type of finite elements (triangles for instance).

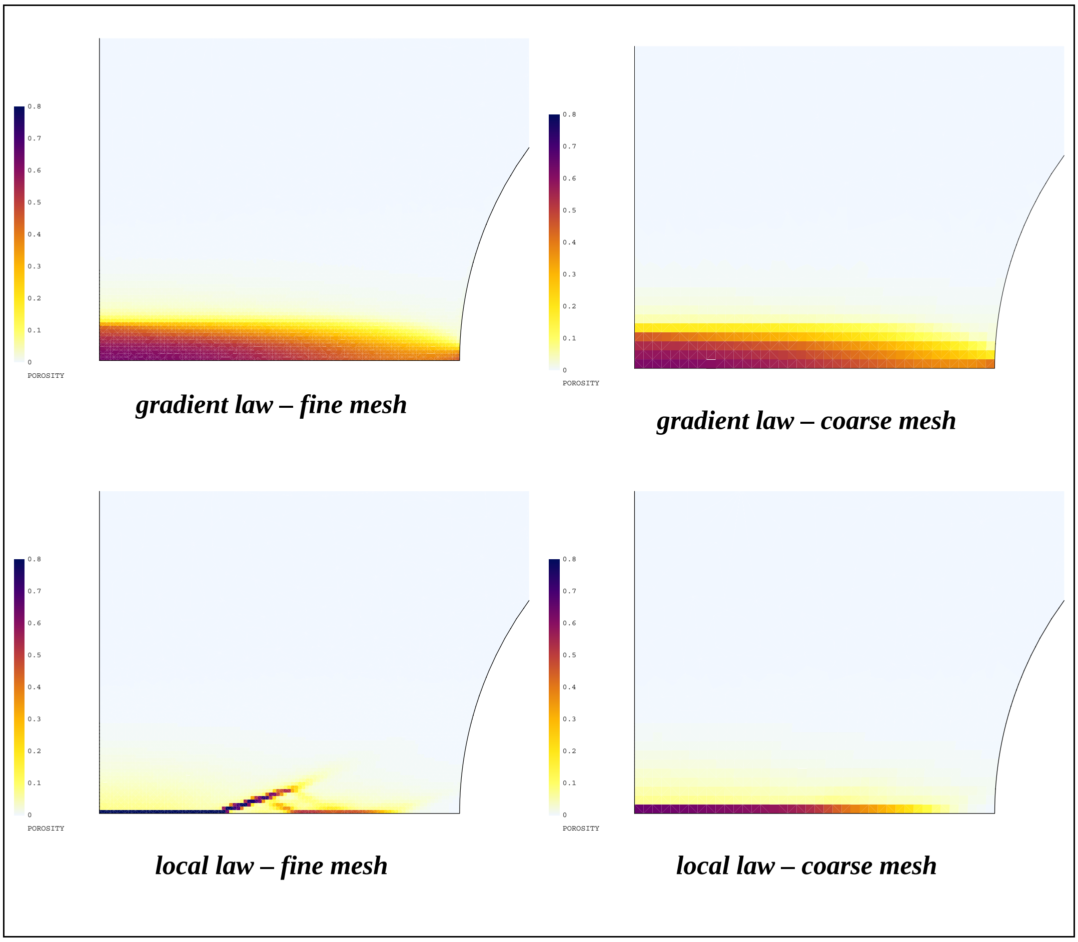

figure 2 – Porosity field (zoom) corresponding to an applied force of 40 000 N

REFERENCES

1. Lorentz E., Andrieux S., A variational formulation for nonlocal damage models, *Int . J. Plas.*, vol. 15, pp. 119-138, 1999.
2. Rousselier G., Finite deformation constitutive relation including ductile fracture damage, in Nemat Nasser (ed.), *3D constitutive relations and ductile fracture*, North Holland, pp. 331-355, 1981.
3. Lorentz E., Cano V., A minimization principle for finite strain plasticity : incremental objectivity and immediate implementation, *Comm. Num. Meth. Engng.*, vol. 18, pp. 851-859, 2002.
4. Simo J.C., Miehe C., Associative coupled thermoplasticity at finite strains : formulation, numerical analysis and implementation, *Comp. Meth. Appl. Mech. Engng.*, vol. 98, pp. 41-104, 1992.
5. Andrieux S., Joussemet M., Lorentz E., A class of constitutive relations with internal variable derivatives : derivation from homogenization, *C. R. Acad. Sci.*, t. 323, série II b, pp. 629-636, 1996.
6. Muhlhaus H.B., Aïfantis E.C., A variational principle for gradient plasticity, *Int. J. Solids Struct.*, vol. 28, no 7, pp. 845-857, 1991.
7. Lorentz E., Benallal A., Gradient constitutive relations : numerical aspects and application to gradient damage, submitted in *Comp. Meth. Appl. Mech. Engng.*