NUMERICAL VALIDATION OF THE SIZE ESTIMATES APPROACH FOR ELECTRICAL CONDUCTORS

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ABSTRACT

This work deals with the so–called *size estimates* approach for detecting inclusions in electrical conductors on the basis of boundary measurements of the current density and of the corresponding voltage. Following the guidelines drawn up in Alessandrini et al. [1] and the experience made in Alessandrini et al. [2], an extended numerical investigation has been performed in order to prove the effectiveness of this approach. The sensitivity with respect to various relevant parameters is also analyzed.
The work is supported by MIUR grant 2003082352 "*Metodi non distruttivi per l'identificazione e la diagnosi di materiali e strutture*".

1 INTRODUCTION

Several applications in the field of geophysical prospection, nondestructive tests and medical imaging lead to the formulation of the following inverse problem: to determine inside an electrical conductor Ω the possible presence of unknown defects, i.e. inclusions made of different material, on the basis of one pair of current and potential boundary measurements. *In mathematical terms*, if u denotes the electrostatic potential defined in the volume Ω of the conductor, one wishes to recover $D \subset\subset \Omega$ in the *Neumann problem*

$$\begin{cases} \operatorname{div}\left((A\chi_{\Omega\setminus D} + \tilde{A}\chi_D)\nabla u\right) = 0 & \text{in } \Omega, \\ A\nabla u \cdot \nu = \varphi & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where A and $\tilde{A}$ denote the conductivity tensors in $\Omega \setminus D$ and in D respectively, ν is the unit exterior normal to $\partial\Omega$, on which a known current field φ is applied, and χ_E denotes the characteristic function of the set E.

The inverse problem under consideration is severely ill–posed, therefore it is significant for the applications to try to estimate, in a stable fashion, some relevant parameters of the unknown defect, such as its volume.

In the paper of Alessandrini et al. [1], under some mild a priori assumptions on the inclusion D, the following constructive upper and lower estimates of the volume of the inclusion D have been obtained:

$$C_1 \left| \frac{W - W_0}{W_0} \right| \leqq |D| \leqq C_2 \left| \frac{W - W_0}{W_0} \right|, \tag{2}$$

where C_1 and C_2 are suitable constants depending on the data of the problem. These estimates are based on the quantities

$$W = \int_{\partial\Omega} u\,\varphi \quad \text{and} \quad W_0 = \int_{\partial\Omega} u_0\,\varphi \tag{3}$$

which represent the power required to maintain the boundary current φ when the inclusion is present (being u the corresponding voltage) and when the inclusion is absent (being u_0 the corresponding voltage), respectively.
It is attractive, from the practical point of view, the use of such estimates in order to evaluate the volume $|D|$ of the inclusion. Indeed, the approach provides that the points $(|\frac{W-W_0}{W_0}|, |D|)$ are confined inside an angular sector delimited by two straight lines passing through the origin. The slopes of these two lines are given by the constants C_1, C_2 appearing in Eq. (2). By referring to Figure 1, we can estimate the confidence interval of $|D|$ starting from the measure of the power gap, once the constants C_1 and C_2 are known.

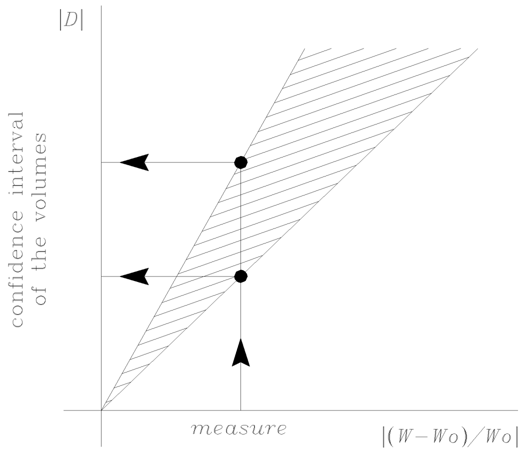

Figure 1: Goal of the size estimates approach.

The concrete evaluation of the constants C_1, C_2 appearing in (2) is a crucial point for real life applications. However, also if the analytical procedure by which they are found is indeed constructive, in practice, it leads to rather pessimistic estimates.
The present work tries to estimate these constants by numerical simulations, following an approach already applied in Alessandrini et al. [2] for the case of elastic inclusions inside elastic bodies.
The numerical experimentation is performend on the basis of a code which can perform the numerical simulations of bi– and three–dimensional bodies made by material with uniform conductivity and subject to prescribed current density on the boundary. The data of the simulations are assumed as simple as possible, but in a way to reproduce the typical experimental settings which frequently occurs in testing of materials.

2 NUMERICAL SIMULATIONS

As already discussed before, we have to estimate the confidence interval of the volume $|D|$ of the defect on the basis of measures of the power gap obtained for all the possible defects. In order to avoid an excessive growing of the numerical tests, we have to focus our attention only on significant tests, aiming to emphasize the main aspects of the problem. In this sense, we will follow the guidelines traced in analyzing the analogous mechanical problem, as faced in Alessandrini et al. [2].

We will perform tests which can indicate how much the position occupied by the defects, its shape and relative conductivity, influence the power gap measure. For making this, we will formulate in the following sections the assumptions adopted for the numerical simulations.

2.1 *Working hypotheses*

We consider both the materials, i.e. the conductor and the defect, as homogeneous materials assuming uniform conductivity. Therefore, it is always possible to express this by a scalar parameter f such that

$$A = f\,\tilde{A}. \tag{4}$$

The influence of f on the power gap will be investigated with $0.1 \leq f \leq 10$ and by maintaining the other parameters representative of the inclusion constant.
From a theoretical point of view, the domain D of the defect have to be contained within the domain Ω of the conductor, at a suitable distance $d_0 > 0$ from the boundary. First of all, we will check this fact, performing some numerical tests with different values of d_0 starting from $d_0 = 0$.
All the numerical simulations use a finite element model of the problem. We will adopt for regular domain the HC formulation for interpolating the electrostatic potential u. The main feature of such an interpolation technique, which can be considered, in bidimensional problems, as a particular case of the Bézier interpolation, consists of its capability in reproducing fields of C^1 smoothness with a computational cost equivalent to a C^0 interpolation. Further details can be found in Aristodemo [3], where the technique has been proposed. For more complex domain, triangular (2D) and tetrahedron (3D) finite elements will be adopted.
Note finally that we will refer to cases of small inclusions, that is $|\Omega|/|D| \leq 6\%$.

2.2 *Generation of the defects*

In order to investigate how the position occupied by the defect influences the power gap, it is more convenient to refer to defects of very regular shape, i.e. squares in 2D or cubes in 3D. In this way, it is also simple to check the sensitivity of the power gap when defects are in the neighborhood of the boundary and therefore to define a minimal distance d_0, and then a subdomain Ω_{d_0}, for which the disturbance of the boundary conditions is negligible.
Successively, within Ω_{d_0} defects of several shape are generated on the basis of the finite element mesh. If one element is considered as the defect of minimum dimension, the possible combinations of this allow to generate very different shapes. However, aiming to generating all the possible combinations, the computational costs could become excessively high. A good choice should lie in using not much fine mesh and a fixed way of generating the shapes, for instance elements with at least a common face. For instance, in analyzing a square plate discretized with a 13×13 mesh we have to generate more than 7000 possible combinations of elements.
However, the number of possible combinations in more complex three–dimensional problems tends to increase exponentially. A way for reducing a priori these combinations is to make some hypothesis on the type of shape of the inclusion. In this sense, the following shape parameters, which measure the “scattering” of the inclusion in two–dimensional problems,

could be significant:

$$\begin{aligned}\text{normalized isoperimetric deficit} &= \frac{P^2 - 4\pi|D|}{4\pi|D|},\\ \text{spread} &= \frac{\lambda_1 + \lambda_2}{|D|^2},\\ \text{elongation} &= \frac{\lambda_2 - \lambda_1}{\lambda_1 + \lambda_2},\end{aligned}$$

where P is the perimeter of the inclusion, and λ_1, λ_2, $0 < \lambda_1 < \lambda_2$ are the eigenvalues of the inertia matrix evaluated with respect to the centroid of the inclusion.

3 CONCLUDING REMARKS

In Alessandrini et al. [1] it was shown how upper and lower bounds of the measure (volume or area) of an unknown defect inside a conductor can be obtained in terms of the power gap. Practical applications requires the evaluation of the constants appearing in these size estimates. On the other hand the complexity of actual problem cut off the use of analytical procedure. Numerical simulations, guided by theoretical results, give a valuable contribution for estimating upper and lower bounds usable in actual applications.

The tests are chosen in order to simulate actual problem. A study on the influence on the upper and lower bound of the significant parameters of the inclusion, such as the position, the shape and the stiffness is useful in order to limit the number of tests. The influence of the boundary conditions are also analyzed to furnish guidelines in experimental test.

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