# STRENGTH PREDICTIONS VIA FINITE FRACTURE MECHANICS

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ABSTRACT
This paper describes a modification to the traditional Griffith energy balance as used in Linear Elastic Fracture Mechanics (LEFM). The modification involves using a finite amount of crack extension instead of an infinitesimal extension when calculating the energy release rate. We propose to call this method Finite Fracture Mechanics (FFM). The finite amount of crack extension ($2L$) is assumed to be a material constant. This modification is extremely useful because it allows LEFM to be used to make predictions in two situations in which it is normally invalid: short cracks and notches. It is shown that accurate predictions can be made of both brittle fracture and fatigue behaviour for short cracks and notches in a range of different materials. The value of $L$ can be expressed as a function of two other material constants: the fracture toughness $K_c$ (or threshold $\Delta K_{th}$ in the case of fatigue) and an inherent strength parameter $\sigma_o$. For the particular cases of fatigue-limit prediction in metals and brittle fracture in ceramics, it is shown that $\sigma_o$ coincides directly with the Ultimate Tensile Strength $\sigma_u$ (or, in fatigue, the fatigue limit), as measured on plain, unnotched specimens. For brittle fracture in polymers and metals, in which larger amounts of plasticity precede fracture, the approach can still be used but $\sigma_o$ takes on a different value, higher than the plain-specimen strength, which can be found from experimental data. Predictions can be made very easily for any problem in which the stress intensity factor, $K$, is known as a function of crack length, e.g. the three point bending test. Furthermore, it is shown that the predictions of this method, FFM, are similar to those of a method known as the Line Method (LM) in which failure is predicted based on the average stress along a line drawn ahead of the crack or notch.

## 1 INTRODUCTION

When dealing with brittle or quasi-brittle materials, two main failure criteria are generally taken into account. The former is a stress criterion: $\sigma = \sigma_u$, i.e. failure takes place if, at least in one point, the stress $\sigma$ reaches the tensile strength $\sigma_u$. The latter is an energetic criterion: $\mathcal{G} = \mathcal{G}_F$; it states that failure happens if the crack driving force $\mathcal{G}$ equals the crack resistance $\mathcal{G}_F$. $\mathcal{G}_F$ is the energy necessary to create unit fracture surface. According to Irwin's relationship, the energetic criterion can be expressed equivalently in terms of stress intensity factor $K$ and toughness $K_c$: $K = K_c$. For the sake of simplicity, in the following only mode I crack propagation will be dealt with.

The stress criterion provides good results for crack-free bodies, whereas the energetic criterion is physically sound for bodies containing a sufficiently large crack. Otherwise both the criteria fail. In fact, the stress criterion provides a null failure load for a body containing a crack, the stress field being singular in front of the crack tip. On the other hand, the energetic criterion provides an infinite failure load for a crack-free body, $K$ being zero in absence of a crack.

The above mentioned criteria therefore work for the extreme cases (i.e. no crack or large crack) but are no longer valid for the intermediate cases, such as, for instance, short cracks or notches (see Carpinteri [1]). In order to overcome this drawback, several failure criteria have been proposed in the literature. For what concerns numerical applications to quasi-brittle materials, we must cite the cohesive crack model, which takes into account both the tensile strength and the fracture energy of the given material. Of course, in order to describe the structural behaviour by the cohesive crack model, a code must be implemented on a computer and the results are obtained numerically. Anyway, easier two-parameter criteria fulfilling the asymptotic cases of strength and toughness can be put forward. To mention just a few, we can cite: the strain energy density criterion by Sih [2], the effective crack model by Karihaloo [3], the size effect model by Bažant [4]. The advantage of all these criteria is the possibility to get analytical results, at least for simple geometries.

An easy way to obtain failure criteria providing the correct asymptotic results (i.e. strength and toughness) will be shown in the following. One can decide whether to modify the strength criterion or the toughness one. In any case, a material length $\Delta$ must be introduced.

For the sake of clarity, we refer to the simple case of a slab with a centre through crack directed along the $x$-axis and $2a$-long. Starting from the strength criterion, we can distinguish between a point-wise stress criterion and an average stress criterion. According to the former, failure is achieved when the stress at a distance $\Delta_{PS}$ from the crack tip reaches the tensile strength $\sigma_u$. According to the latter, failure is achieved when the average stress ahead of the crack tip over a $\Delta_{LS}$-long segment reaches the critical value $\sigma_u$. Or, alternatively, when the stress resultant over a segment of length $\Delta_{LS}$ in front of the crack tip reaches the critical value $\sigma_u\ \Delta_{LS}$. In formulae, respectively:

$$\sigma_y\left(a+\Delta_{PS}\right)=\sigma_u\ , \qquad \int_a^{a+\Delta_{LS}}\sigma_y\left(x\right)\mathrm{d}x=\sigma_u\Delta_{LS} \tag{1}$$

The subscripts PS and LS stand for "point-wise stress" and "line stress" criteria. Of course both these criteria provide $\sigma = \sigma_u$ for a crack-free specimen under a tensile load. On the other hand, the values of the material lengths must be determined imposing $K = K_c$ for relatively large crack ($\Delta$<<$a$). In such a case, the asymptotic stress field $\sigma_y = K/\sqrt{2\pi x}$ can be used; it provides:

$$\Delta_{LS}=4\Delta_{PS}=\frac{2}{\pi}\left(\frac{K_c}{\sigma_u}\right)^2 \tag{2}$$

Obviously, in order to provide meaningful results in the intermediate cases, the stress field to be inserted in the stress failure criteria (1) must be the exact one and not the asymptotic one. This means, for instance in the case of a centre through crack, to use Westergaard's solution. When the exact solution is not available, the stress field can be obtained numerically by a finite element analysis. The stress failure criteria (1) are sometimes referred to as point method (PM) and line method (LM).

Let's now turn our attention to energy failure criteria. In their derivation, we can follow the same way as done for stress criteria. In fact, even in this case, we can distinguish between a point-wise energy criterion and an average energy criterion. According to the former, failure is achieved when the crack driving force for a crack of length ($a$ + $\Delta_{PT}$) reaches the crack resistance $\mathcal{G}_F$. According to the latter, failure is achieved when the energy available in a crack advance equal to $\Delta_{LT}$ reaches the critical value $\mathcal{G}_F\ \Delta_{LT}$. Recalling Irwin's relationship, we get the two formulae:

$$K\left(a+\Delta_{PT}\right)=K_c\ , \qquad \int_a^{a+\Delta_{LT}}K^2\left(a\right)\mathrm{d}a=K_c^2\Delta_{LT} \tag{3}$$

The subscripts PT and LT stand for "point-wise toughness" and "line toughness" criteria. Of course both these criteria provide $K = K_c$ for a specimen containing a large crack. On the other hand, the values of the material lengths must be determined imposing $\sigma = \sigma_u$ for a vanishing crack ($a$ = 0). In such a case, considering only through crack for two-dimensional geometries, the asymptotic value for the stress intensity factor is given by $K=F_1\sigma\sqrt{\pi a}$ where $F_1$ is a dimensionless factor equal to 1 for centre cracks and equal to 1.12 for edge cracks. Substituting this expression in eqns (3) provides:

$$\Delta_{\mathrm{LT}} = 2\Delta_{\mathrm{PT}} = \frac{2}{\pi}\left(\frac{K_{\mathrm{c}}}{F_1\sigma_{\mathrm{u}}}\right)^2 \tag{4}$$

Obviously, in order to provide meaningful results, the values of the stress intensity factor to be inserted in the energy failure criteria (3) must be the exact one and not the asymptotic one. Anyway, note that in the case of a centre or an edge through crack in an infinite slab, the expression $K = F_1\sigma\sqrt{\pi a}$ is not only asymptotic: it is valid for any $a$ value. Therefore the computation of the critical stress for such easy cases is straightforward, whereas the stress failure criteria require Westergaard's solution. For more complex geometries, the analytical solution is usually not available; nevertheless, the $K(a)$ function can be obtained from the stress intensity factor handbooks or from LEFM software codes. Finally, note that the point-wise toughness criterion is sometimes called equivalent LEFM.

While the point-wise criteria lack a physical background, the physical meaning of the average stress and energy criteria is clear: fracture does not propagate continuously but by finite crack extensions, at least at the first step. Particularly, for the energetic failure criterion, it is assumed that the energy release is not continuous but discrete; i.e. it happens by finite steps: the minimum energy quantity necessary to have crack propagation is $\mathcal{G}_{\mathrm{F}}\,\Delta_{\mathrm{LT}}$. This is the reason why we call this approach Finite Fracture mechanics (FFM, or Quantized Fracture Mechanics, i.e. QFM, especially if this approach is applied at nanoscale [5]). Furthermore, according to the authors' opinion, FFM is a novel criterion, whereas LM, PM and equivalent LEFM are well known in the literature [6,7]. Only Seweryn et al. [8] has proposed recently a similar failure criterion.

The aim of the present paper is to show some applications of the FFM: our opinion is that is very versatile, being able to catch the transition between short and long cracks, the transition between cracks and notches and the size effect. In the remaining part of the paper, the finite crack extension (4) will be denoted by $2L$. Defining $\sigma_0$ as the "inherent strength" of the material (usually equal to $F_1$ times the ultimate tensile strength – UTS – see below), therefore:

$$L = \frac{1}{\pi}\left(\frac{K_{\mathrm{c}}}{\sigma_0}\right)^2 \tag{5}$$

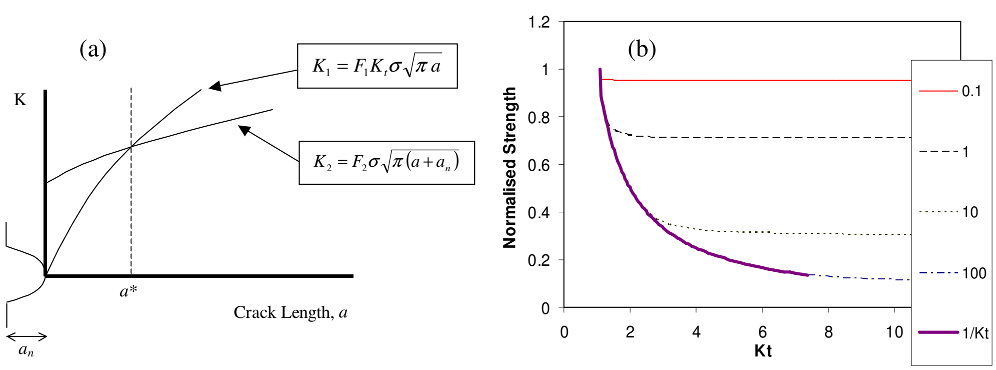


Figure 1: Edge notch scheme (a) and FFM strength prediction for blunt and sharp notches varying the crack length, i.e. the ratio $a_n/L$ (b)

## 2 NOTCH ANALYSIS

Two parameters affect the behaviour of notches in comparison to that of cracks: root radius and notch angle. Here we will consider the effect of root radius for a notch of zero angle (i.e. a U-shaped notch), with length $a_n$ and root radius $\rho$. Fig.1a shows a through-thickness edge notch, though in fact the derivation will apply to any shape (e.g. an elliptical cavity) provided $a_n$ is much smaller than the dimensions of the body itself. The loading again takes the form of a remote, normal, tensile stress $\sigma$. Fig.1a shows how the stress intensity, $K$, increases for a crack growing from the root of the notch. The form of this increase is complex and difficult to represent analytically, so we will use a simplified form which is commonly used in notch/crack analysis. When the crack is relatively small its stress intensity is approximately given by $K_1$, where:

$$K_1 = F_1 K_t \sigma \sqrt{\pi a} \tag{6}$$

Here $K_t$ is the elastic stress concentration factor of the notch (equal to the maximum stress at the notch root divided by $\sigma$) and $F_1$ is a constant which depends on the geometry of the notch and crack. When the crack is relatively large its stress intensity is given by $K_2$, where:

$$K_2 = F_2 \sigma \sqrt{\pi \left(a + a_n\right)} \tag{7}$$

Here $F_2$ is the geometry factor for a crack of total length $(a+a_n)$. The two solutions cross at $a$=$a^*$, where:

$$a^* = a_n \frac{F_2^2}{(F_1^2 K_t^2 - F_2^2)} \tag{8}$$

As before, we consider the energy changes consequent on a finite amount ($2L$) of crack growth. There are two possible cases. The former one is $2L < a^*$; in this case only eqn (6) is needed:

$$\sigma_f = \frac{K_c}{F_1 K_t \sqrt{\pi L}} \tag{9}$$

The full effect of the stress concentration factor ($\sigma_f = \sigma_u / K_t$) is experienced if $\sigma_0 = F_1 \sigma_u$. See below for a discussion about this point.

The latter case is $2L > a^*$; accordingly, the change in strain energy is given by:

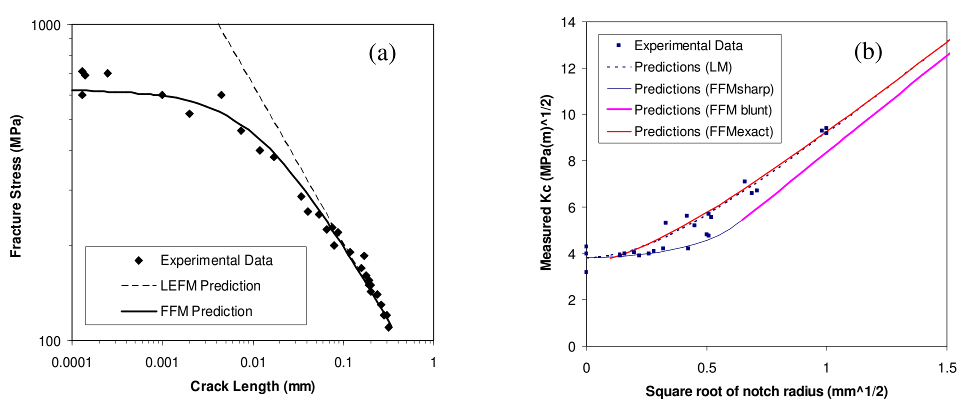


Figure 2: FFM predictions. Experimental data for the strength of Silicon Carbide (a) and measured $K_c$ as a function of notch root radius for Alumina (b)

$$\Delta W = \int_0^{a^*} \frac{K_1^{\,2}}{E} da + \int_{a^*}^{2L} \frac{K_2^{\,2}}{E} da \tag{10}$$

Equating $\Delta W$ to $\mathcal{G}_c(2L)$ gives a prediction for $\sigma_f$ for the notch:

$$\sigma_f = \frac{1}{F_2} \frac{K_c}{\sqrt{\pi Q}}, \qquad \text{where } Q = a_n - \left[ \frac{F_2^2}{2} \frac{a_n^2}{(F_1^{\,2} K_t^2 - F_2^2) 2L} \right] + L \tag{11}$$

The parameter $Q$ has three terms. The first term, $a_n$, dominates in cases of long, sharp cracks ($a_n >> L$ and $K_t$ tending to infinity). The second term (in square brackets) modifies the equation to account for notches; the third term, $L$, controls the size effect, giving a reduced strength which tends to $\sigma_o$ as the length of the crack or notch tends to zero. Fig.1b shows how the result changes with normalised notch length (plotting only the valid parts of the curves): for small notches the strength tends to a constant value at all $K_t$, which approaches unity as $a_n/L$ approaches zero.

## 3 COMPARISON WITH EXPERIMENTAL DATA

This section presents experimental data taken from various sources in the literature.

Fig.2a shows typical data on the fracture strength of a ceramic material - Silicon Carbide - tested by Kimoto et al [9]. The FFM theory predicts the results very well, over the whole range of crack lengths from long cracks (which conform to standard LEFM predictions) to very short cracks which have no significant effect on specimen strength.

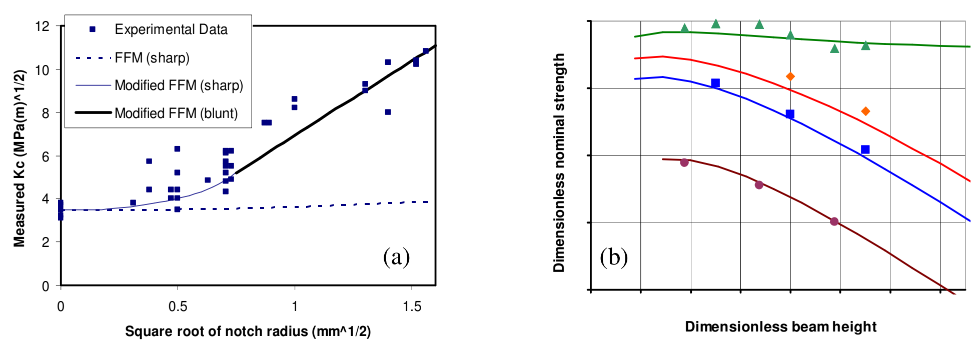


Figure 3: FFM predictions and experimental data for brittle fracture in polycarbonate (a); bi-logarithmic plot for strengths of TPB high strength concrete notched specimens (b): from top to bottom, curves relate to a relative crack depth equal to 0, 0.05, 0.10, 0.30.

Fig.2b shows some data due to Tsuji et al [10] on the measured fracture toughness (i.e. the $K_c$ value that would be measured in an experiment using a notched specimen, assuming that the notch was the same as a crack) of ceramic specimens containing cracks and notches, as a function of root radius. The material is alumina. The LM and FFM criteria both give reasonable predictions, with the FFM (blunt and sharp solutions) forming a lower bound to the data and giving a slight underestimate of strength at high $\rho$. This is due to the simplification used in estimating $K_1$ and $K_2$, which tends to give an overestimate of the strain energy. The FFM prediction can be made more accurately in a particular case by using the appropriate $F$ factors for stress intensity, taken from stress intensity factor handbooks. We carried out this analysis for the data on Fig.2b, giving the prediction line labeled “FFM exact”. This shows that FFM and the LM give almost identical predictions, though there are some slight differences. This exact FFM calculation can only be done knowing the *F* factor as a function of crack length for each specimen geometry; thus we will continue to use the simplified, general form below.

Up to now we have been able to make accurate predictions using eqn (5) to calculate *L*, assuming that the inherent strength $\sigma_o$, is equal to the plain-specimen strength $\sigma_u$ (i.e. the UTS or fatigue limit of the material), incorporating crack shape effects through the *F* factor. We have found this to be successful in predicting many sets of data on the fatigue strengths of metals and the fracture strengths of ceramics. This is very convenient because it means that no new constants are required in order to use the theory. However when we considered brittle fracture in polymers and metals, we found that, although accurate predictions could be made, it was necessary to use a different value of *L*. Fig. 3a illustrates this for data on a polymer (Polycarbonate) reported by Tsuji et al [10]. Both the original and modified predictions are shown in Fig.3a, indicating that the value of *L* calculated using the UTS was too large. Accurate predictions were obtained using a lower value, *L*'; it implies a value of $\sigma_o$ higher than the UTS by a factor of about 4 (labelled 'modified FFM' in Fig. 3a). This is less satisfactory from a prediction point of view, as the value of *L*' can only be known by finding a best fit to the experimental data. This implies that data is required for two different notches – ideally a crack and a relatively blunt notch. Even so the predictive capacity of the theory is still very high.

Finally, a comparison with the experimental data obtained by Karihaloo et al. [11] for three point bending tests of high strength concrete specimens has been performed. Tests relate to different relative notch depths and to different specimen sizes. In Fig. 3b we plotted the strength vs. the specimen size in a bi-logarithmic diagram. The FFM prediction fits reasonably well to the experimental data, moreover catching the change of the concavity of the size effect curves when passing from un-notched to notched specimens. Note that the two best fit parameters, i.e. the material strength and toughness, are the same for all the curves plotted in Fig. 3b.

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