AN INTERACTION INTEGRAL METHOD FOR COMPUTATION OF T-STRESS ALONG THE FRONTS OF GENERAL NON-PLANAR CRACKS IN THREE-DIMENSIONS

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ABSTRACT

An interaction integral method is described for extracting the T-stress distribution along the fronts of general, non-planar cracks in three dimensions. The three-dimensional auxiliary fields that are required in the method are defined by extruding two-dimensional fields along the three-dimensional crack front. The two-dimensional fields are those associated with a point load applied at the tip of an semi-infinite crack in an infinitely extended solid. After defining the auxiliary fields, the T-stress distribution along the crack front is be obtained by evaluating the domain form of the interaction integral–a volume integral surrounding the point of interest on the crack front. In order to evaluate the integral, the finite element method is employed to obtain the stress, strain and displacement fields in the cracked body. In order to assess the accuracy of the present method, the problem of a thick plate with an inclined center crack is considered, and the present numerical results for T-stress as a function of crack inclination angle are compared with the analytical solution. To demonstrate the utility of the method, the problems of a lens-shaped crack and a warped elliptical crack embedded in a cylinder and subjected to remote tension are considered and the results are discussed.

1 INTRODUCTION

The asymptotic expansion of the near-tip stress fields is composed of the well-known $1/\sqrt{r}$ singular terms, a non-singular term, and higher order non-singular terms that vanish as r approaches the crack tip, Williams [1]. The first non-singular term that appears in the near-tip fields is commonly called the T-stress. Knowledge of the T-stress is important, because it has been shown to play a significant role on crack growth under mixed-mode loading conditions, and also on crack path stability under pure mode I loading conditions.

The interaction integral method, also known as the M-integral method, has proven to be a useful tool for extracting mixed-mode stress intensity factors in both 2-D and 3-D fracture problems, see Shi and Asaro [2], and Gosz and Moran [3]. In the interaction integral method, auxiliary fields are superposed on top of actual fields that come from the solution to the boundary value problem of interest. The J-integral associated with the superposed fields, J^s, can be expressed in terms of the J-integral associated with the actual fields, J, the J-integral associated with the auxiliary fields, J^{aux}, and an interaction integral, i.e., $J^s = J + J^{aux} + I$. The interaction integral can be expressed in terms of desired crack-tip parameters through a judicious choice of auxiliary fields.

In the present paper, for the purpose of extracting the T-stress distribution along non-planar, three-dimensional crack fronts, three-dimensional auxiliary fields are constructed by extruding two-dimensional near-tip fields along the three-dimensional crack front. The two-dimensional fields are taken to be those that arise from a point force applied at the tip of a semi-infinite crack in an infinitely extended body. With this choice of auxiliary fields, the T-stress can be obtained simply by evaluating the interaction integral. The present study is conducted within the framework of linearized elasticity theory, and small strain kinematics is assumed throughout.

For computational reasons, it is advantageous to recast the interaction integral into an equivalent domain integral. In the present paper, the domains of integration are tubular domains surrounding the points of interest on the crack front. For a given cracked geometry and external loading, the boundary value problem can be solved using any suitable numerical technique. In the present paper, the finite element method is employed. The crack-tip parameters are extracted in a post-processing step. Unstructured meshes of 4-node tetrahedral elements are used to obtain the stress, strain and displacement fields in the cracked body. In the present method, the geometry of the crack surface and crack front can either be described mathematically, or they can be approximated, for example, by Bezier patches and curves respectively. The Bezier patch and Bezier curve representations of the crack surface and crack front allow the gradients of certain vector quantities that appear in the domain form of the interaction integral to be readily calculated.

To outline the remainder of the paper, in the next section the crack surface and crack front geometry are defined, and a local (orthogonal curvilinear) crack-tip coordinate system is set up in which to define the auxiliary fields. In Section 3, the interaction integral is defined. The auxiliary fields which appear in the integrand of the interaction integral are defined in Section 4 for extraction of mixed-mode stress intensity factors and T-stress. The domain form of the interaction integral is derived in Section 5, and details are provided for evaluating the domain integral as a post-processing step in the finite element method. Some numerical results are presented in Section 6. In particular, the accuracy of the present method is assessed by considering the problem of a thick plate with an inclined center crack. The numerical results are compared with the analytical solution by Smith *et al.* [6] as a function of crack angle. The second example is that of a lens-shaped crack embedded in an infinite cylinder subjected to hydrostatic tensile loading. The results are compared to the analytical solution by Martynenko and Ulitko [4]. The final example is that of a warped-elliptical crack surface embedded in a cylinder subjected to remote tensile loading. This is an example for which, to the author's knowledge, there is no analytical solution. The distribution of mixed-mode stress intensity factors and T-stress are obtained and the results are discussed.

2 CRACK SURFACE GEOMETRY

To begin, consider the crack surface shown in figure 1. The geometry of the crack surface can be described mathematically as

$$f(x_1, x_2, x_3) = 0, \tag{1}$$

where x_1, x_2, and x_3 form the axes of a Cartesian coordinate system. A position vector $\mathbf{x}(s)$ defines the location of point s which lies on the crack front. The unit tangent vector, $\mathbf{T}$, to the crack front at point s is given as

$$\mathbf{T} = \frac{d\mathbf{x}}{ds} / \left| \frac{d\mathbf{x}}{ds} \right|, \tag{2}$$

where ds represents an infinitesimal movement along the crack front. We now let the vector $\mathbf{b}$ be defined as the unit outward normal to the crack surface at point s. It can be expressed as

$$\mathbf{b} = \vec{\nabla} f / \left| \vec{\nabla} f \right|. \tag{3}$$

A right handed system of three mutually perpendicular base vectors can now be constructed by letting $\mathbf{c} = \mathbf{T} \times \mathbf{b}$. We note that the unit vector $\mathbf{c}$ is perpendicular to the crack front and lies in the local tangent plane to the crack front at point s. The direction of each unit vector $\mathbf{T}$, $\mathbf{b}$, and $\mathbf{c}$ changes

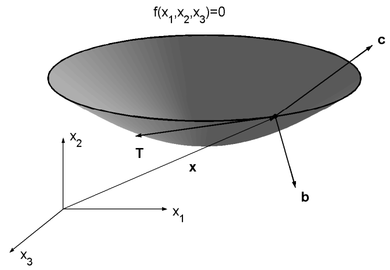

Figure 1: Three-dimensional crack surface

continuously as point s moves along the crack front.

3 INTERACTION INTEGRALS

In this section, we define the interaction integrals for extracting mixed-mode stress intensity factors and T-stress along the fronts of three-dimensional, nonplanar crack surfaces. The J-integral, $J(s)$, evaluated at some point s along the crack front, can be written as

$$J(s) = \lim_{\Gamma\to 0} c_l(s) \int_{\Gamma(s)} (W\delta_{lj} - \sigma_{ij} u_{i,l}) n_j d\Gamma \tag{4}$$

where $\Gamma(s)$ is a contour lying in the same plane as the unit vectors $\mathbf{b}$ and $\mathbf{c}$ that surrounds point s. Here in eqn (4), σ_{ij} is the Cauchy stress, W is the strain energy density, and $u_{i,l}$ are the components of the displacement gradient tensor. The quantity n_j represents the components of the unit outward normal to the contour Γ, and $c_l(s)$ are the components of the unit vector $\mathbf{c}$. The fields that appear inside the J-integral come from the solution to the actual boundary value problem of interest. These will be referred to as the actual fields.

If we superpose auxiliary fields on top of the actual fields, the J-integral associated with the combination of both fields, J^s, can be expressed in terms of the J-integral (4) associated with the actual fields, J, the J-integral, J^{aux}, associated with the auxiliary fields, and the interaction integral, I, as follows:

$$J^s = J + J^{aux} + I. \tag{5}$$

The interaction integral, I, (also known as the M-integral) is defined as

$$I(s) = \lim_{\Gamma\to 0} c_l(s) \int_{\Gamma(s)} (\sigma_{ik}\epsilon^{aux}_{ik}\delta_{lj} - \sigma_{ij}u^{aux}_{i,l} - u_{i,l}\sigma^{aux}_{ij}) n_j d\Gamma, \tag{6}$$

where σ^{aux}_{ij}, ϵ^{aux}_{ij}, and u^{aux}_i are the auxiliary stress, strain, and displacement fields defined in the next section.

4 DEFINITION OF THE AUXILIARY FIELDS

For the purpose of defining the auxiliary fields, it is convenient to set up a local coordinate system whose origin is located at point s on the crack front. The local $x_1^{'}, x_2^{'}, x_3^{'}$ axes are chosen such that the $x_1^{'}$ axis is always aligned with the vector $\mathbf{c}$, the $x_2^{'}$ axis always points in the $-\mathbf{b}$ direction, and the $x_3^{'}$ axis is parallel to the vector $\mathbf{T}$ so as to form a right-handed coordinate system. We note that as point s moves along the three-dimensional crack front, the orientation of the local axes continuously changes so that they remain parallel to the vectors $\mathbf{T}$, $\mathbf{b}$ and $\mathbf{c}$.

4.1 *Auxiliary fields for extracting mixed-mode stress intensity factors*

For any point s that lies on the crack front, the auxiliary stress, strain, and displacement fields are defined to be the plane and anti-plane crack-tip fields. It is convenient to express the components of these fields in the local (primed) coordinate system. It is also convenient to introduce the polar coordinates r and θ. The coordinate r is the distance from the crack tip to a point lying in the $\mathbf{b}$-$\mathbf{c}$ plane. The angle θ is measured counterclockwise from the $x_1^{'}$ axis to the point of interest.

In the definitions below, the indices i and j range from 1 to 2, and refer to components in the primed coordinate system. The auxiliary stress field is defined as

$$\sigma_{ij}^{aux} = \frac{K_I^{aux}}{\sqrt{2\pi r}} f_{ij}^{I}(\theta) + \frac{K_{II}^{aux}}{\sqrt{2\pi r}} f_{ij}^{II}(\theta) \tag{7}$$

$$\sigma_{i3}^{aux} = \frac{K_{III}^{aux}}{\sqrt{2\pi r}} f_{i}^{III}(\theta) \tag{8}$$

$$\sigma_{33}^{aux} = \nu(\sigma_{11}^{aux} + \sigma_{22}^{aux}). \tag{9}$$

Here in eqns (7)-(9), the parameter ν is Poisson's ratio, and K_I^{aux}, K_{II}^{aux}, and K_{III}^{aux} are the stress intensity factors associated with the auxiliary fields. The quantities $f_{ij}^{I}(\theta)$, $f_{ij}^{II}(\theta)$, and $f_{i}^{III}(\theta)$ are the angular functions associated with the near-tip fields.

The auxiliary displacement components are defined as

$$u_1^{aux} = \frac{1}{8\mu}\sqrt{\frac{2r}{\pi}}\left[K_I^{aux} f^{I}(\theta) + K_{II}^{aux} f^{II}(\theta)\right] \tag{10}$$

$$u_2^{aux} = \frac{1}{8\mu}\sqrt{\frac{2r}{\pi}}\left[K_I^{aux} g^{I}(\theta) + K_{II}^{aux} g^{II}(\theta)\right] \tag{11}$$

$$u_3^{aux} = \frac{1}{\mu}\sqrt{\frac{2r}{\pi}}\left[K_{III}^{aux} g^{III}(\theta)\right] \tag{12}$$

where μ is the shear modulus.

Finally, the components of the auxiliary strain field are defined as

$$\epsilon_{ij}^{aux} = \frac{1}{2}\left(u_{i,j}^{aux} + u_{j,i}^{aux}\right) \tag{13}$$

$$\epsilon_{13}^{aux} = \frac{1}{2}\frac{\partial u_3^{aux}}{\partial x_1^{'}} \tag{14}$$

$$\epsilon_{23}^{aux} = \frac{1}{2}\frac{\partial u_3^{aux}}{\partial x_2^{'}} \tag{15}$$

$$\epsilon_{33}^{aux} = 0. \tag{16}$$

We note that in the definitions (13)-(16), all derivatives with respect to the coordinate $x_3^{'}$ are zero. For a given point s on the crack front, and for any given point r and θ, the auxiliary fields do not change with respect to a movement in the direction of the tangent vector, **T**, which is always parallel to the $x_3^{'}$ axis.

It turns out that when the actual fields (with stress intensity factors K_I, K_{II} and K_{III}) and the auxiliary fields (with stress intensity factors K_I^{aux}, K_{II}^{aux}, and K_{III}^{aux}) defined in the manner above, are substituted into the interaction integral (6), the result (in the limit as the contour is shrunk onto point s) is

$$I(s) = \frac{2(1-\nu^2)}{E}\left[K_I K_I^{aux} + K_{II} K_{II}^{aux}\right] + \frac{1}{\mu} K_{III} K_{III}^{aux}. \tag{17}$$

Thus, the stress intensity factors associated with the actual fields can be obtained by evaluating the interaction integral. For example, the stress intensity factor K_I can be obtained by setting $K_I^{aux} = 1$ and $K_{II}^{aux} = K_{III}^{aux} = 0$. In this example we would obtain

$$K_I(s) = \frac{E}{2(1-\nu^2)} I(s). \tag{18}$$

4.2 *Auxiliary fields for extracting the T-stress*

For the purpose of extracting the T-stress at points along the crack front, we choose the auxiliary displacement fields to be those associated with a point force, F, applied at the tip of a semi-infinite crack in an infinitely extended body. Hence, the auxiliary displacement components referred to the primed coordinate system are defined as

$$u_1^{aux} = -\frac{F(1+\kappa)}{8\pi\mu} \ln\frac{r}{d} - \frac{F}{4\pi\mu}\sin^2(\theta), \tag{19}$$

$$u_2^{aux} = -\frac{F(\kappa-1)}{8\pi\mu}\theta + \frac{F}{4\pi\mu}\sin(\theta)\cos(\theta). \tag{20}$$

$$u_3^{aux} = 0. \tag{21}$$

where κ is Kolosov's constant (κ=$3-4\nu$ for the case of plane strain), and d is a characteristic length from point s to another point p that lies on the $x_1^{'}$ axis.

The auxiliary strain field is defined as

$$\epsilon_{ij}^{aux} = \frac{1}{2}(u_{i,j}^{aux} + u_{j,i}^{aux}), \tag{22}$$

where the indices i and j range from 1 to 2 only. All of the out of plane auxiliary strain components are taken to be zero, i.e., $\epsilon_{13}^{aux} = \epsilon_{23}^{aux} = \epsilon_{33}^{aux} = 0$. The auxiliary stress field is obtained from the auxiliary strain field using Hooke's law for the case of plane strain.

When the auxiliary fields are defined to be those associated with a point load F applied at the tip of a semi-infinite crack in an infinitely extended and linearly elastic body, it can be shown that the interaction integral (6) reduces to

$$I(s) = \frac{TF}{E}, \tag{23}$$

where T is the T-stress, see Paulino and Kim [5] for more details. Hence,

$$T(s) = \frac{E}{F} I(s). \tag{24}$$

5 DOMAIN FORM OF THE INTERACTION INTEGRAL

In order to facilitate the numerical computation of the interaction integrals defined in the previous section, it is advantageous to recast the contour integrals into equivalent domain integrals. The equivalent domain form of the interaction integral is obtained by defining an appropriate test or weighting function and applying the divergence theorem. In the paper, the derivation of the domain integrals for extraction of mixed-mode stress intensity factors will be provided. In addition, some important aspects regarding the numerical evaluation of the domain integrals will be discussed. Of particular importance is the fact that we are dealing with volume integrals (whose domains extend significantly outside the near-tip region). Outside of the near-tip region, it turns out that the auxiliary strain fields described above are not compatible with the auxiliary displacement fields. That is, the auxiliary strain fields are not the symmetric gradient of the auxiliary displacement fields. It also turns out that the auxiliary stress fields are not in equilibrium outside of the near-tip region. This lack of compatibility and equilibrium gives rise to some extra terms in the domain integrals which require careful consideration. This holds true for both evaluation of mixed-mode stress intensity factors and T-stress.

6 NUMERICAL RESULTS

In the numerical results section we will consider three example problems. The first problem will be that of a thick plate with an inclined center crack. The plate is subjected to uniform tensile loading. The numerical results for mixed-mode stress intensity factors and T-stress are reported for a variety of crack orientations. The numerical results for T-stress are compared to the analytical result by Smith *et al.* [6]. The second problem is that of a lens-shaped crack embedded in a cylinder and subjected to all around tension. The mixed-mode stress intensity factors using the interaction integral method have been shown to compare favorably with the analytical solution by Martynenko and Ulitko [4] in Gosz and Moran [3]. In the present paper, the T-stress distribution is also obtained and the results are discussed. The final problem is that of a warped elliptical crack front embedded in a block subjected to remote tension. Numerical results for the T-stress distribution along the non-planar crack front are obtained and the results are discussed.

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