A CONSISTENT DESCRIPTION OF THE UNILATERAL EFFECT OF ORTHOTROPIC DAMAGE ON ELASTIC PROPERTIES

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ABSTRACT
The existing anisotropic damage models accounting for the unilateral damage effect are suffering from the drawback that the thermodynamic potential associated to a given thermodynamic state is not unique. Starting from this observation, this work proposes a physically consistent and coordinate-free formulation when the damage state can be characterized by a second-order symmetric tensor. Four cases are distinguished according as the multiplicity of the eigenvalues of this damage tensor, and the unilateral damage effect on elastic properties is correctly modeled so that the thermodynamic potential for each thermodynamic state is unique.

1 INTRODUCTION

Materials in which microcracks are embedded behave differently in tension and compression. In particular, the tensile stiffness is generally lower than the compressive one. This macroscopic dissymmetry results from the unilateral contacts of microcracks: under tension the microcracks are open and the stresses cannot be transmitted across the microcrak lips while under compression the microcracks are closed and the stresses are continuous across the microcrack lips. Modeling of the unilateral effect of anisotropic damage is a lastingly standing issue of damage mechanics. Although the first general attempt to take into account the damage activation and deactivation goes back to the work of Ortiz [1], the relevant problem has not completely been solved up to now.

Two main difficulties are involved in modeling of the unilateral effect of anisotropic damage. The first is due to the requirement that the damaged stress-strain relation be continuous across any damage activation-deactivation separating surface. The second comes from the requirement that the strain- or stress-energy be unique for any given thermodynamic state. Among all existing models dealing with the unilateral effect of anisotropic damage, a few ones [2-3] satisfy the stress-strain continuity condition but no one verifies the thermodynamic potential uniqueness requirement. This important observation has been recently made by Cormery and Welemane [4] (see also Carol and Willam [5]).

The objective of the present work is to propose a consistent model capable of correctly describing the unilateral effect of orthotropic damage on elastic properties. This work is initially motivated by the observation of Cormery and Welemane [4] and its objective is achieved by modifying a model owing to

Chaboche [2] and using the results recently obtained by He [6] who treated an acoustic problem.

2 DAMAGE VARIABLE AND SPECTRAL DECOMPOSITION

A symmetric second order tensor $\underline{\underline{D}}$ is chosen as the single damage internal variable, $\underline{\underline{D}}$ can be expressed in its principal axes according to the spectral decomposition:

$$\begin{aligned}\underline{\underline{D}} &= d_1 \underline{n}^{(1)} \otimes \underline{n}^{(1)} + d_2 \underline{n}^{(2)} \otimes \underline{n}^{(2)} + d_3 \underline{n}^{(3)} \otimes \underline{n}^{(3)} \\ &= d_1 \underline{\underline{N}}^{(1)} + d_2 \underline{\underline{N}}^{(2)} + d_3 \underline{\underline{N}}^{(3)}\end{aligned} \tag{1}$$

In this first part, we summarize some results recently obtained by He [6] in another context. This approach permits firstly to explicitly calculate the eigenvalues of $\underline{\underline{D}}$ and identify the multiplicity of each eigenvalue in terms of the principal invariants of $\underline{\underline{D}}$, and secondly to specify the eigen-projection operators of $\underline{\underline{D}}$.

With no loss of generality, the three eigenvalues of $\underline{\underline{D}}$ are ordered so that:

$$d_1 \geq d_2 \geq d_3 \tag{2}$$

Firstly, we introduce the principal invariants of $\underline{\underline{D}}$:

$$I_1 = \mathrm{tr}\underline{\underline{D}}, \quad I_2 = \frac{1}{2}\left[\left(\mathrm{tr}\underline{\underline{D}}\right)^2 - \mathrm{tr}\underline{\underline{D}}^2\right], \quad I_3 = \det \underline{\underline{D}} \tag{3}$$

Next, we define the following four isotropic functions of $\underline{\underline{D}}$:

$$f = f(I_1, I_2, I_3) = I_1^2 - 3I_2\,, \tag{4.a}$$

$$g = g(I_1, I_2, I_3) = 27I_3^2 + 4I_2^3 - I_1^2 I_2^2 + 4I_1^3 I_3 - 18 I_1 I_2 I_3\,, \tag{4.b}$$

$$h = h(I_1, I_2, I_3) = \frac{2}{27} I_1^3 - \frac{1}{3} I_1 I_2 + I_3\,, \tag{4.c}$$

$$\omega = \omega(I_1, I_2, I_3) = \frac{1}{3}\cos^{-1}\frac{27 I_1^3 - 9 I_1 I_2 + 27 I_3}{2\left(I_1^2 - 3I_2\right)^{3/2}} \text{ with } 0 \leq \omega \leq \frac{\pi}{3}. \tag{4.d}$$

Note that the angle ω was firstly introduced by Lode [7].

The main results obtain by He [6] can be summarized as follows, distinguish four cases according to the multiplicity of each eigenvalue of $\underline{\underline{D}}$:

FIRST CASE: $d_1 > d_2 > d_3$

This case is true if end only if $f \neq 0,\ \ g \neq 0$ and the three eigenvalues of $\underline{\underline{D}}$ are given by:

$$\begin{aligned} d_1 &= \frac{1}{3} I_1 + \frac{2}{3} (I_1^2 - 3I_2)^{\frac{1}{2}} \cos\omega \\ d_2 &= \frac{1}{3} I_1 + \frac{2}{3} (I_1^2 - 3I_2)^{\frac{1}{2}} \cos\left(\frac{2}{3}\pi - \omega\right) \\ d_3 &= \frac{1}{3} I_1 + \frac{2}{3} (I_1^2 - 3I_2)^{\frac{1}{2}} \cos\left(\frac{2}{3}\pi + \omega\right) \end{aligned} \tag{5}$$

The damage tensor $\underline{\underline{D}}$ is given by $\underline{\underline{D}} = d_1 \underline{\underline{N}}^{(1)} + d_2 \underline{\underline{N}}^{(2)} + d_3 \underline{\underline{N}}^{(3)}$ where:

$$\underline{\underline{N}}^{(1)} = \frac{\left(\underline{\underline{D}} - d_2 \underline{\underline{\delta}}\right) . \left(\underline{\underline{D}} - d_3 \underline{\underline{\delta}}\right)}{(d_1 - d_2)(d_1 - d_3)}, \tag{6.a}$$

$$\underline{\underline{N}}^{(2)} = \frac{\left(\underline{\underline{D}} - d_1 \underline{\underline{\delta}}\right) . \left(\underline{\underline{D}} - d_3 \underline{\underline{\delta}}\right)}{(d_2 - d_1)(d_2 - d_3)}, \tag{6.b}$$

$$\underline{\underline{N}}^{(3)} = \frac{\left(\underline{\underline{D}} - d_1 \underline{\underline{\delta}}\right) . \left(\underline{\underline{D}} - d_2 \underline{\underline{\delta}}\right)}{(d_3 - d_1)(d_3 - d_2)} \tag{6.c}$$

where $\underline{\underline{\delta}}$ is the second order identity tensor.

SECOND CASE: $d_1 = d_2 = d_3 = d$

This case takes place only if $f = 0$. It follows that $\underline{\underline{D}} = d\underline{\underline{\delta}}$.

THIRD CASE: $d_1 > d_2 = d_3$

This case occurs if and only if $f \neq 0,\ \ g = 0,\ \ h > 0$ and we have:

$$d_1 = \frac{1}{3} I_1 + \frac{2}{3} \sqrt{I_1^2 - 3I_2}\ ,\ d_2 = d_3 = \frac{1}{3} I_1 - \frac{1}{3} \sqrt{I_1^2 - 3I_2} \tag{7}$$

The damage tensor $\underline{\underline{D}}$ is given by $\underline{\underline{D}} = d_1 \underline{\underline{N}}^{(1)} + d_2 (\underline{\underline{\delta}} - \underline{\underline{N}}^{(1)})$ where :

$$\underline{\underline{N}}^{(1)}=\frac{1}{\sqrt{I_1^2-3I_2}}\left[\underline{\underline{D}}-\frac{1}{3}\left(I_1-\sqrt{I_1^2-3I_2}\right)\underline{\underline{\delta}}\right] \tag{8}$$

<u>FOURTH CASE:</u> $d_1=d_2>d_3$

This case occurs if and only if $f\neq 0,\ \ g=0,\ \ h<0$ and we have:

$$d_1=d_2=\frac{1}{3}I_1+\frac{1}{3}\sqrt{I_1^2-3I_2}\ ,\ d_3=\frac{1}{3}I_1-\frac{2}{3}\sqrt{I_1^2-3I_2} \tag{9}$$

The damage tensor $\underline{\underline{D}}$ is given by $\underline{\underline{D}}=d_1(\underline{\underline{\delta}}-\underline{\underline{N}}^{(3)})+d_3\underline{\underline{N}}^{(3)}$ where :

$$\underline{\underline{N}}^{(3)}=\frac{1}{\sqrt{I_1^2-3I_2}}\left[\frac{1}{3}\left(I_1+\sqrt{I_1^2-3I_2}\right)\underline{\underline{\delta}}-\underline{\underline{D}}\right] \tag{10}$$

3 FORMULATION OF CHABOCHE [2]

This formulation proposed by Chaboche [2] constitutes a general framework and can be applied to any macroscopic damage model. The thermodynamic potential is expressed as:

$$w(\underline{\underline{\varepsilon}},\underline{\underline{D}})=\frac{1}{2}\underline{\underline{\varepsilon}}:\mathbf{C}:\underline{\underline{\varepsilon}} \tag{11}$$

where the elasticity tensor $\mathbf{C}$ depends on the state $(\underline{\underline{\varepsilon}},\underline{\underline{D}})$ and is defined by:

$$\mathbf{C}=\widetilde{\mathbf{C}}+\eta\sum_{i=1}^{3}H\left(-\underline{\underline{\varepsilon}}:\underline{\underline{N}}^{(i)}\right)\left(\underline{\underline{N}}^{(i)}\otimes\underline{\underline{N}}^{(i)}\right):\left(\mathbf{C}^0-\widetilde{\mathbf{C}}\right):\left(\underline{\underline{N}}^{(i)}\otimes\underline{\underline{N}}^{(i)}\right) \tag{12}$$

In this expression, H represents the Heavyside function, η $(0\leq\eta\leq 1)$ is a parameter which characterizes the intensity of the elastic moduli recovery. $\mathbf{C}^0$ is the undamaged elasticity tensor and $\widetilde{\mathbf{C}}$ denotes the elasticity tensor when all microcracks are active (open).

Assuming that the undamaged material is isotropic and linear elastic, $\mathbf{C}^0$ is given by:

$$\mathbf{C}^0=\lambda_0\underline{\underline{\delta}}\otimes\underline{\underline{\delta}}+2\mu_0\underline{\underline{\delta}}\overline{\underline{\otimes}}\underline{\underline{\delta}} \tag{13}$$

where $\underline{\underline{\delta}}$ is the second order identity tensor and λ_0 and μ_0 are the Lame coefficients of the undamaged materials. In Eq. (13), the following tensor products of two second-order tensors $\underline{\underline{A}}$ and $\underline{\underline{B}}$ is used : $\left(\underline{\underline{A}}\,\overline{\otimes}\,\underline{\underline{B}}\right)_{ijkl}=\frac{1}{2}\left(A_{ik}B_{jl}+A_{il}B_{jk}\right)$.

The tensor $\widetilde{\mathbf{C}}$ can, for example, being defined by:

$$\widetilde{\mathbf{C}}=\lambda_0\left(\underline{\underline{\delta}}+\underline{\underline{D}}\right)\otimes\left(\underline{\underline{\delta}}+\underline{\underline{D}}\right)+2\mu_0\left(\underline{\underline{\delta}}+\underline{\underline{D}}\right)\overline{\underline{\otimes}}\left(\underline{\underline{\delta}}+\underline{\underline{D}}\right) \tag{14}$$

As shown by Cormery and Welemane [4], w given by eqn (11) is not an admissible thermodynamic potential due to the no uniqueness of the set $\left(\underline{\underline{N}}^{(1)},\underline{\underline{N}}^{(2)},\underline{\underline{N}}^{(3)}\right)$ in second, third and fourth cases.

According to these remarks, the formulation of Chaboche [2] is modified using the results summarized in section 2. The four cases are successively studied.

<u>First case:</u> $d_1>d_2>d_3$

In this case, the formulation proposed by Chaboche [2] is admissible and we have:

$$\mathbf{C}=\widetilde{\mathbf{C}}+\eta\sum_{i=1}^{3}H\left(-\underline{\underline{\varepsilon}}:\underline{\underline{N}}^{(i)}\right)\left(\underline{\underline{N}}^{(i)}\otimes\underline{\underline{N}}^{(i)}\right):\left(\mathbf{C}^0-\widetilde{\mathbf{C}}\right):\left(\underline{\underline{N}}^{(i)}\otimes\underline{\underline{N}}^{(i)}\right) \tag{15}$$

where the eigen-projection operators of $\underline{\underline{D}}$, $\underline{\underline{N}}^{(i)}$ $\left(i=1,2\text{ or }3\right)$, are defined by eqns (6).

<u>Second case:</u> $d_1=d_2=d_3=d$

This case corresponds to an isotropic damage. Activation-deactivation condition is defined by $\mathrm{tr}\underline{\underline{\varepsilon}}=0$ and the corresponding damaged elasticity tensor is given by:

$$\mathbf{C}=\widetilde{\mathbf{C}}+\eta H\left(-\mathrm{tr}\underline{\underline{\varepsilon}}\right)\frac{1}{9}\left(\underline{\underline{\delta}}\otimes\underline{\underline{\delta}}\right):\left(\mathbf{C}^0-\widetilde{\mathbf{C}}\right):\left(\underline{\underline{\delta}}\otimes\underline{\underline{\delta}}\right) \tag{16}$$

<u>Third case:</u> $d_1>d_2=d_3$

In this case, corresponding to transverse isotropic damage, two activation-deactivation conditions must be defined. For the no-repeated eigenvalue (d_1), this condition is defined by $\underline{\underline{\varepsilon}}:\underline{\underline{N}}^{(1)}=0$ with $\underline{\underline{N}}^{(1)}$ defined by eqn (8) and for the double eigenvalue ($d_2=d_3$), we have $\underline{\underline{\varepsilon}}:\underline{\underline{N}}^{\perp}=0$ with $\underline{\underline{N}}^{\perp}=\underline{\underline{\delta}}-\underline{\underline{N}}^{(1)}$. The corresponding damaged elasticity tensor is expressed as:

$$\mathbf{C} = \widetilde{\mathbf{C}} + \eta \left\{ \begin{array}{l} H\left(-\underline{\underline{\varepsilon}} : \underline{\underline{N}}^{(1)}\right)\left(\underline{\underline{N}}^{(1)} \otimes \underline{\underline{N}}^{(1)}\right) : \left(\mathbf{C}^0 - \widetilde{\mathbf{C}}\right) : \left(\underline{\underline{N}}^{(1)} \otimes \underline{\underline{N}}^{(1)}\right) \\ H\left(-\underline{\underline{\varepsilon}} : \underline{\underline{N}}^{\perp}\right)\left(\underline{\underline{N}}^{\perp} \otimes \underline{\underline{N}}^{\perp}\right) : \left(\mathbf{C}^0 - \widetilde{\mathbf{C}}\right) : \left(\underline{\underline{N}}^{\perp} \otimes \underline{\underline{N}}^{\perp}\right) \end{array} \right\} \tag{17}$$

<u>FOURTH CASE:</u> $d_1 = d_2 > d_3$

This case is identical with the previous one and we obtained:

$$\mathbf{C} = \widetilde{\mathbf{C}} + \eta \left\{ \begin{array}{l} H\left(-\underline{\underline{\varepsilon}} : \underline{\underline{N}}^{(3)}\right)\left(\underline{\underline{N}}^{(3)} \otimes \underline{\underline{N}}^{(3)}\right) : \left(\mathbf{C}^0 - \widetilde{\mathbf{C}}\right) : \left(\underline{\underline{N}}^{(3)} \otimes \underline{\underline{N}}^{(3)}\right) \\ H\left(-\underline{\underline{\varepsilon}} : \underline{\underline{N}}^{\perp}\right)\left(\underline{\underline{N}}^{\perp} \otimes \underline{\underline{N}}^{\perp}\right) : \left(\mathbf{C}^0 - \widetilde{\mathbf{C}}\right) : \left(\underline{\underline{N}}^{\perp} \otimes \underline{\underline{N}}^{\perp}\right) \end{array} \right\} \tag{18}$$

where $\underline{\underline{N}}^{\perp} = \underline{\underline{\delta}} - \underline{\underline{N}}^{(3)}$ with $\underline{\underline{N}}^{(3)}$ given by eqn (10).

4 FINAL REMARKS

The formulation suggested makes it possible to ensure the uniqueness of the thermodynamic potential. The modifications suggested can be applied to the model proposed by Dragon and Halm [3]. In addition, the major advantage of this approach lies in the fact that the eigenvalues and the eigen-projection operators of $\underline{\underline{D}}$ are explicitly expressed according to its principal invariants.

5 REFERENCES

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