A CRITICAL STATE TWO-SURFACE MICROPOLAR PLASTITCITY MODEL FOR SANDS

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ABSTRACT
A critical state two-surface plasticity model is extended to incorporate the behavior of granular soils in the post failure regime by using a micropolar framework and incorporating the couple stress terms as well as force-stress terms in the yield function. The proposed formulation provides a unified constitutive model for modeling the stress-strain response of sands for a wide range of confining stresses and soil densities at the pre-failure and post failure regimes.

1 INTRODUCTION

Recent investigations have demonstrated the use of micropolar plasticity in modeling post failure response of geomaterials (e.g., Manzari, 2004). Various geomechanics problems ranging from bearing capacity of shallow foundations to stability of slopes and retaining walls where shear is the dominant mode of failure have been shown to be suitable for the use of micropolar elastoplastic constitutive models. The entire spectrum of the material stress-strain response from very small elastic deformations to the nonlinear inelastic response before a peak strength and the post peak response which involves strain localization can be modeled with a unified micropolar elastoplastic model without facing the usual difficulties encountered in the solution of geomechanics problems using standard elastoplastic models. Despite this capability, micropolar plasticity has been applied mainly to classical plasticity models with the exceptions of a few hypoplastic constitutive models. On the other hand, it has long been recognized that stress-strain response of soil is best described within a critical state framework where the tendency of soil to dilate or contract is determined with respect to its current confining pressure and density and possibly its degree of anisotropy. A recently proposed critical state two-surface plasticity model (Manzari and Dafalias, 1997; Dafalias and Manzari, 2004) has been shown to reproduce the drained and undrained stress-strain responses of sands under a wide range of confining stresses and densities in both monotonic and cyclic loading regimes.

Given the versatility of the above mentioned two-surface plasticity model in modeling the pre-failure stress-strain behavior and the desirable features of micropolar plasticity in modeling post failure response, this paper presents a critical state two-surface micropolar plasticity model for sands. The model consists of a single yield surface that undergoes kinematic hardening in the force-stress space. The plastic potential is indirectly defined through a dilatancy coefficient, which is dependent on the current stress state and soil density. Following the premise of bounding surface plasticity, the evolution law for the tensor of internal variables describing kinematic hardening is defined in terms of the distance between the current stress state and its image on an appropriately defined bounding surface (outer surface). The same technique is used to determine dilatancy coefficient as a function of the distance between current stress state and its image on a dilatancy surface. The evolution laws for bounding surface and dilatancy surface are defined in such a way that at a critical state, both surfaces collapse onto a unique critical state surface.

The proposed formulation presents a unified constitutive model for modeling the stress-strain response of sands for a wide range of confining stresses and soil densities at both pre-failure and post failure regimes.

2 MICROPOLAR CONTINUUM

2.1 Kinematics

At any material point of the continuum, we consider both a displacement and a rotation vector denoted by **u** and ϕ, respectively. The so-called Cosserat micro-rotation **R** relates the current state of a triad of orthonormal directions attached to each material point to its initial state (Forest, 2001), i.e.

$$R_{ij} = \delta_{ij} - \Gamma_{ijk}\phi_k \tag{1}$$

where δ_{ij} is Kronecker delta and $\boldsymbol{\Gamma}$ is the permutation tensor. The associated Cosserat deformation $\boldsymbol{\varepsilon}$ and curvature tensor $\boldsymbol{\kappa}$ are written as

$$\begin{aligned} \varepsilon_{ij} &= u_{j,i} - \Gamma_{ijk}\phi_k \\ \kappa_{ij} &= \phi_{j,i} \end{aligned} \tag{2}$$

It is clear that in the absence of rotation vector, ϕ, classical continuum mechanics is recovered.

2.2 Balance laws

It is assumed that the transfer of interaction between two particles of the continuum through a surface element $\mathbf{n}dS$ (**n** is the normal to the surface element) occurs by means of both a traction vector $\mathbf{t}dS$ and a moment vector, $\mathbf{m}dS$. Surface forces and couples are represented by the generally non-symmetric force-stress and couple-stress tensors $\boldsymbol{\sigma}$, and $\boldsymbol{\mu}$, respectively (Forest 2001).

$$\begin{aligned} \mathbf{t} &= \boldsymbol{\sigma}.\mathbf{n} \\ \mathbf{m} &= \boldsymbol{\mu}.\mathbf{n} \end{aligned} \tag{3}$$

The axioms of balance of linear momentum and moment of momentum require that the following equations hold:

$$\sigma_{ji,j} + f_i = \rho \ddot{u}_i$$
$$\mu_{ji,j} + \Gamma_{ikl}\sigma_{kl} + \varsigma_i = I\ddot{\phi}_i \qquad (4)$$

in which $\mathbf{f}$ and ς, respectively, represent vectors of body forces and body couples, ρ is mass density and I denotes the isotropic rotational inertia. In a specific boundary value problem, appropriate additional boundary conditions should be defined to accompany Eqs. (3) & (4).

3 CONSTITUTIVE EQUATIONS

Solution of a boundary value problem that is governed by Eqs. (4), requires proper constitutive equations linking the deformation and curvature tensors to the force and couple-stresses. Here we will use an elastoplastic constitutive model that is suitable for granular materials. Details of this constitutive model are outlined in the following subsections.

3.1 Additive Decomposition for Small Perturbations

Here we use the usual additive decomposition of deformation and curvature tensors that are applicable in small perturbation regime. The deformation and curvature tensors are decomposed to an elastic part denoted by a superscript "e" and a plastic part indicated by a superscript "p".

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^e + \dot{\boldsymbol{\varepsilon}}^p$$
$$\dot{\boldsymbol{\kappa}} = \dot{\boldsymbol{\kappa}}^e + \dot{\boldsymbol{\kappa}}^p \qquad (5)$$

3.2 Elastic Response

The general isotropic elastic relationships for a micropolar continuum is given as (Nowacki, 1986):

$$\dot{\sigma}_{ij} = \lambda \dot{\varepsilon}^e_{kk}\delta_{ij} + 2\mu\dot{\varepsilon}^e_{(ij)} + 2\mu_c\dot{\varepsilon}^e_{\{ij\}}$$
$$\dot{\mu}_{ij} = \alpha\dot{\kappa}^e_{kk}\delta_{ij} + 2\beta\dot{\kappa}^e_{(ij)} + 2\gamma\dot{\kappa}^e_{\{ij\}} \qquad (6).$$

Where $\dot{\varepsilon}_{(i,j)}$ and $\dot{\varepsilon}_{\{i,j\}}$ respectively denote the symmetric and skew symmetric parts of $\dot{\varepsilon}_{ij}$, while $\dot{\kappa}_{(i,j)}$ and $\dot{\kappa}_{\{i,j\}}$ represent the symmetric and skew symmetric parts of $\dot{\kappa}_{ij}$. In addition to the usual Lame constants, four additional elasticity moduli appear in Eqs. (6). To simplify elastic relations used in the later development, we consider: $\alpha = \beta = 0$ and

$$\gamma = \frac{4\mu\mu_c}{\mu + \mu_c} l_1^2 \qquad (7)$$

where l_1 is a characteristic length. Hence, a simplified form of Eq. (6) is obtained as follows:

$$\dot{\sigma}_{ij} = \lambda\dot{\varepsilon}^e_{kk}\delta_{ij} + (\mu + \mu_c)\dot{\varepsilon}^e_{ij}$$
$$\dot{\mu}_{ij} = \gamma\dot{\kappa}^e_{ij} \qquad (8)$$

in which the first equation reduces to Hooke's law for isotropic elasticity in the absence of Cosserat rotation (i.e. when strain tensor is symmetric).

In accordance with the critical state two-surface plasticity model for sands (Dafalias and Manzari, 2004) the standard Lame parameters λ and μ are obtained from the following equations, which indicate the pressure dependence nature of elastic moduli, reflecting a key characteristic of geomaterials.

$$\mu = \mu_0 \, p_{at} \frac{(2.97-e)^2}{1+e} (\frac{p}{p_{at}})^{1/2}; \quad \lambda = \frac{2\nu}{1-2\nu}\mu \qquad (9)$$

where p_{at} is atmospheric pressure, p is mean effective stress, e is current void ratio, ν is Poisson's ratio, and μ_0 is a model constant.

3.3 Yield Function

The yield function is assumed to have the following form:

$$f = [\tilde{J}_2]^{1/2} - \sqrt{2/3}\,pm = 0 \qquad (10)$$

in which $\tilde{J}_2$ is now defined as

$$\tilde{J}_2 = a_1 \tilde{s}_{ij}\tilde{s}_{ij} + a_2 \tilde{s}_{ij}\tilde{s}_{ji} + b_1 \mu_{ij}\mu_{ij} + b_2 \mu_{ij}\mu_{ji} \qquad (11)$$

with $\tilde{s}_{ij} = s_{ij} - p\alpha_{ij}$, where $s_{ij} = \sigma_{ij} - p\delta_{ij}$ is the deviatoric (force-) stress tensor and α is an unsymmetric deviatoric tensor defined as a tensor of internal variables. Parameter m indicates the size of the yield surface that is assumed to be constant in the present formulation. Parameters a_1, a_2, b_1, and b_2 may be calculated on the basis of micromechanical consideration of particles displacements and rotations in a granular medium (Vardoulakis and Sulem, 1995).

3.4 Flow Rule

A non-associative flow rule, similar to the flow rule proposed in Manzari and Dafalias (1997), is adopted as follows:

$$\begin{aligned} \dot{\boldsymbol{\varepsilon}}^p &= \langle L \rangle \mathbf{R}^\sigma \\ \dot{\boldsymbol{\kappa}}^p &= \langle L \rangle \mathbf{R}^\mu \end{aligned} \qquad (12)$$

where:

$$L = \frac{1}{K_p}(\frac{\partial f}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} + \frac{\partial f}{\partial \boldsymbol{\mu}} : \dot{\boldsymbol{\mu}}) \qquad (13)$$

in which K_p is the plastic modulus and

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} = \mathbf{n}^{\sigma} - \frac{1}{3}(N^{\sigma})\mathbf{I}$$
$$\frac{\partial f}{\partial \boldsymbol{\mu}} = \mathbf{n}^{\mu} - \frac{1}{3}(N^{\mu})\mathbf{I} \qquad (14)$$

Hence we define $\mathbf{R}^{\sigma}$ and $\mathbf{R}^{\mu}$ as follows:

$$\mathbf{R}^{\sigma} = \mathbf{n}^{\sigma} + \frac{1}{3}D\mathbf{I}$$
$$\mathbf{R}^{\mu} = \frac{\partial f}{\partial \boldsymbol{\mu}} = \mathbf{n}^{\mu} - \frac{1}{3}N^{\mu}\mathbf{I} \qquad (15)$$

Hence, the dilatancy ratio, D, is related to the plastic volumetric strain through the following equation:

$$\dot{\varepsilon}_v^p = \langle L \rangle D \qquad (16)$$

in which D is defined as

$$D = A_d(\boldsymbol{\alpha}_{\theta}^d \text{-} \boldsymbol{\alpha}):\mathbf{n}^{\sigma} \qquad (17)$$

where $\boldsymbol{\alpha}_{\theta}^d$ is defined similar to the equations proposed by Manzari and Dafalias (1997) and Dafalias and Manzari (2004), i.e.

$$\boldsymbol{\alpha}_{\theta}^d = \sqrt{2/3}\,[g(\theta,c)\,M\exp(n^d\psi) - m]\,\mathbf{n}^{\sigma} \qquad (18)$$
$$A_d = A_0\ (1 + \langle \mathbf{z} : \mathbf{n}^{\sigma} \rangle) \qquad (19)$$

A key feature of Equation (18) is the presence of state parameter, $\psi = e - e_c$, which makes dilatancy (D) a function of the distance between current void ratio (e) and the critical state void ratio (e_c) corresponding to the current mean effective stress (p). **z** is the fabric-dilatancy tensor which is the key element in capturing the response of sand in reverse loading (Dafalias and Manzari, 2004). θ is a modified Lode angle defined as:

$$\theta = \frac{1}{3}\sin^{-1}\left(\frac{-3\sqrt{3}}{2}\frac{J_{3s}}{J_{2s}^{3/2}}\right); \qquad -\frac{\pi}{6} \le \theta \le \frac{\pi}{6} \qquad (20)$$

where the terms J_{2s} and J_{3s} are the second and third invariant of the symmetric part of deviatoric (force-) stress tensor, s_{ij}, are defined as:

$$J_{2s} = \frac{1}{2}s_{(ij)}\,s_{(ij)}; \quad J_{3s} = \frac{1}{3}s_{(ij)}\,s_{(jk)}\,s_{(ki)} \qquad (21)$$

Here again $s_{(i,j)}$ denotes the symmetric part of deviatoric stress tensor, s_{ij}, i.e.

$$s_{(ij)} = \frac{1}{2}(s_{ij} + s_{ji}) \qquad (22)$$

3.5 Hardening law

The evolution law for the back stress ratio tensor, α, is defined in the same way as proposed in Manzari and Dafalias (1997), i.e.

$$\dot{\boldsymbol{\alpha}} = \langle L \rangle\ (2/3)\ h\ (\boldsymbol{\alpha}_\theta^b - \boldsymbol{\alpha}) \qquad (23)$$

where:

$$h = b_0 / (\boldsymbol{\alpha} - \boldsymbol{\alpha}_{in}) : \mathbf{n}^\sigma \qquad (24)$$

with the following relations for $\mathbf{b}_0$ and $\boldsymbol{\alpha}_\theta^b$:

$$b_0 = G_0\, h_0\, (1 - c_h\, e)(p / p_{at})^{-1/2} \qquad (25)$$

$$\boldsymbol{\alpha}_\theta^b = \sqrt{2/3}[\, g(\theta, c)\, M \exp(\mp n^b \psi) - m]\, \mathbf{n}^\sigma \qquad (26)$$

where h_0, G_0, c_h and n^b are model parameters. Moreover, we use the relation proposed in Dafalias and Manzari (2004) to define the evolution of fabric-dilatancy tensor, $\mathbf{z}$:

$$\dot{\mathbf{z}} = -c_z\ \langle -d\varepsilon_v^p \rangle\ (z_{max}\ \mathbf{n}^\sigma + \mathbf{z}) \qquad (27)$$

where c_z and Z_{maz} are model parameters.

4. REFERENCES

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5. ACKNOWLEDGEMENT

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