STATICALLY ADMISSIBLE STRESS RECOVERY FOR CRACK PROBLEMS

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ABSTRACT
A statically admissible stress recovery (SAR) method is introduced. The basis functions used in each patch to fit the stresses at sampling points (e.g., quadrature points) obtained by the finite element method (FEM) satisfy the equilibrium equations, and traction conditions on the local boundary. The continuity of the stress between patches is enforced using the moving least squares (MLS) technique. The results for a typical mixed mode crack show the promise of SAR. As a general technique, SAR can be used with traditional FEs as well as the recently introduced meshless methods. Together with SAR the extended FEM is likely to be powerful for complicated crack problems where the exact local solutions are not available, e.g. cohesive cracks.

1 INTRODUCTION

Stress recovery techniques have attracted great attention in the development of the finite element method (FEM) because they improve the accuracy of the FE stress results by an *a posteriori* treatment, and play a crucial role in the *a posteriori* error estimation.

Most stress recovery techniques include two steps. Firstly, they fit the stresses at sampling points (e.g., quadrature points) by a polynomial of proper order via least squares (LS) technique within a patch of elements for which the number of sampling points can be taken as greater than the number of parameters in the polynomial. Secondly, the stresses at a node are obtained by averaging the fitted results from those patches which include this node to make the final stress field continuous within the considered domain. Two widely used techniques are the superconvergent patch recovery technique (SPR) introduced by Zienkiewicz and Zhu [1] and the recovery technique based on the equilibrium of patches (REP) introduced by Boroomand and Zienkiewicz [2]. SPR works for problems when the stresses sampled at certain points in an element possess the superconvergent property (i.e., they converge at the same rate as displacement at these points; at all other points the convergence will be slower). REP works when supercovergent points do not exist.

On the other hand, Tabbara et al. [3] obtained a continuous stress field directly with the moving least squares (MLS) technique. However, as in the SPR and REP, the recovered stresses are not statically admissible, i.e., they neither meet the equilibrium condition within the domain nor the traction conditions on the local boundaries.

Extensive studies have shown that it is essential to enforce the equilibrium and satisfy the traction conditions on the local boundary to improve the accuracy of the stresses (Wu and Cheung [4]; Karihaloo and Xiao [5]). Enhanced SPR incorporating equilibrium and boundary conditions has been studied by Wiberg et al. [6] and Blacker and Belytschko [7]. However, Babuška et al. [8] showed that the enhanced SPR generally gives much worse robustness index performance than the original SPR, especially on irregular elements assembled near boundaries. The reason seems to lie in the discontinuity of stresses that are irrelevant to the use of direct LS across the boundary of patches. The stress field is thus not really statically admissible.

In this paper, a statically admissible stress recovery scheme (SAR) will be introduced. Basis functions that meet the equilibrium in the domain as well as traction conditions on the local boundaries are used in the recovery process for each patch. The continuity of the stress between patches is enforced by using the MLS technique instead of the direct LS technique. A single edge cracked plate under end shearing (SECPS) will be studied to demonstrate the potential of the method.

2 STATICALLY ADMISSIBLE STRESS RECOVERY

The MLS interpolant $s^h(x)$ of the stress $s(x)$ is defined in the domain Ω by

$$\sigma^h(x) = p(x)\beta(\tilde{x}) \tag{1}$$

where $p(x)$ are complete basis functions in the spatial coordinates x, which satisfy equilibrium in the domain and meet traction conditions on the local boundaries. The coefficients $\beta(\tilde{x})$ are also functions of x and are obtained by minimizing the following weighted L_2 norm:

$$J(\beta) = \sum_{I}^{n} w(\tilde{x} - x_I)[p(x_I)\beta(\tilde{x}) - \hat{\sigma}_I]^T [p(x_I)\beta(\tilde{x}) - \hat{\sigma}_I] \tag{2}$$

where n is the number of points in the domain of influence (DOI) of $\tilde{x}$ for which the weight function $w(\tilde{x} - x_I) \neq 0$, and $\hat{\sigma}_I$ is the value of s (x) at sampling point (e.g. quadrature point) $x = x_I$. The statically admissible basis functions of stress components are normally coupled. Therefore the recovery is implemented for the whole stress vector instead of each component individually.

The weight function used in the current study is a fourth order spline function

$$w(\tilde{x} - x_I) = \begin{cases} 1 - 6(d_i/r_i)^2 + 8(d_i/r_i)^3 - 3(d_i/r_i)^4 & 0 \le d_i \le r_i \\ 0 & d_i \ge r_i \end{cases} \tag{3}$$

in which $d_i = |\tilde{x} - x_I|$ is the distance from node x_I to point $\tilde{x}$. The symbol r_i is the radius of the DOI. Detailed information for implementing the MLS technique can be found in Tabbara et al. [3].

For two-dimensional problems, a first order complete self-equilibriated stress field meeting homogeneous equilibrium equations has 7 *b*s; while a second order complete stress field has 12 *b*s. Self-equilibriated stress fields satisfying also conditions on one traction-free side can be found in Xiao and Karihaloo [5]. Such a second order complete stress field has 6 *b*s. Self-equilibriated stress fields satisfying conditions on two adjacent traction-free sides can be obtained from the stress field in Xiao and Karihaloo [5] after satisfying the homogeneous equilibrium requirements and reducing the number of stress parameters. Such a second order complete stress field has 5 *b*s.

The body forces as well as inhomogeneous tractions on the boundary can be enforced directly by appending suitable special terms to the stress fields.

3 NUMERICAL RESULTS FOR THE SECPS

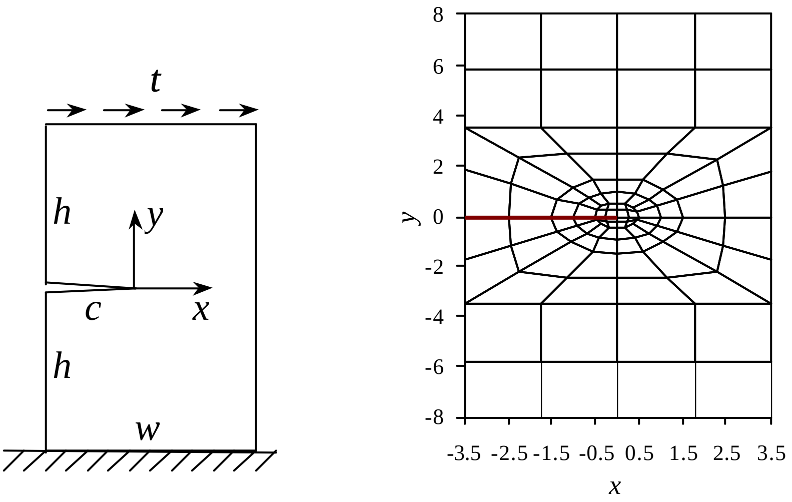
The geometrical parameters of the SECPS (Fig. 1a) considered here are: $c = 3.5m$, $w = 7m$, $h = 8m$; and the shear stress $t = 1\ N/m^2$. Young's modulus $E = 10^5\ N/m^2$, Poisson's ratio $n = 0.25$. A state of plane stress is considered and the thickness is assumed to be unity. The FE mesh is shown in Fig. 1b. All nodes on the bottom line ($y = -8m$) are fixed both in x- and y-directions. The 4-node plane isoparametric element Q4 and hybrid stress element PS (Pian and Sumihara [9]) are used. 2×2 Gauss quadrature is employed for their formulation. Stresses at these quadrature points are used in the recovery process. Nodal stresses on circles with radii $r = 0.5$ and 1.0 surrounding the crack tip (Fig. 1b), and on the line of extension of the crack are compared in Figs. 2-4. The nodal stresses for Q4 elements are obtained by bilinear extrapolation of the stresses at quadrature points; the nodal stresses for PS elements are obtained directly from their trial stresses. The stresses corresponding to the first 15 terms (of each mode) of the Williams expansion obtained by Xiao et al. [10] using a 17-node hybrid crack element (HCE) are also included as highly accurate reference solutions. In the stress recovery process, a linearly complete self-equilibriated stress field with 7 *b*s is used for all nodes in elements not adjacent to the traction-free boundary; a second order statically admissible stress field is used for nodes inside elements adjacent to the boundary (6 bs for one traction-free side; 5 bs for two adjacent traction-free sides). For nodes on the common boundary of these two kinds of element, both stress fields are used to test the continuity of the recovered stresses. The radius r_i of the DOI used by MLS are as follows: for circle r = 0.5 / 1.0, r_i =0.35 / 0.5 for the two nodes adjacent to the crack faces, and r_i =0.2 / 0.4 for other points; for the 5 nodes on the line of extension of the crack, r_i = 0.2, 0.2, 0.4, 0.5, and 1 are used, respectively. r_i for nodes inside (near the boundary) the domain is chosen close to the smallest side (diagonal) of surrounding elements. Thus, there are 8 – 11 quadrature points in the DOI of each point.

In Figs 2-4, a line segment or represents stresses within an element inside (above) the circle (line); a line segment or represents stresses within an element outside (below) the circle (line); represents the recovered stresses; represents HCE results.

In Figs. 2-4, the nodal stresses of Q4 obtained by extrapolation and PS from its trial stress fields are obviously discontinuous along the boundary of elements. PS predicts more accurate stresses, especially close to the crack tip (Fig. 2). After averaging nodal stresses among surrounding elements, both elements give quite satisfactory stresses along the line of extension of the crack away from the crack tip (Fig. 4). However, as the mesh used (Fig. 1b) is very coarse and seriously distorted, the angular distribution, which is important for predicting the direction of crack propagation, is very different with the reference solution. The stresses recovered by SAR are however continuous and more accurate for both elements especially close to the crack tip.

4 CONCLUSIONS AND DISCUSSION

The SAR using MLS technique and statically admissible basis functions is very promising. It evidently improves the accuracy of the stresses; the recovered stresses are continuous although different basis functions are used in different regions. This is achieved by using MLS and choosing the basis functions for adjacent regions with similar number of parameters (instead of the order of completeness) when similar number of sampling points are used in the recovery process.



(a) (b)

Fig. 1: (a) Geometry and loading conditions for the SECPS; (b) FE mesh with 96 quadrilateral elements giving a total of 115 nodes.

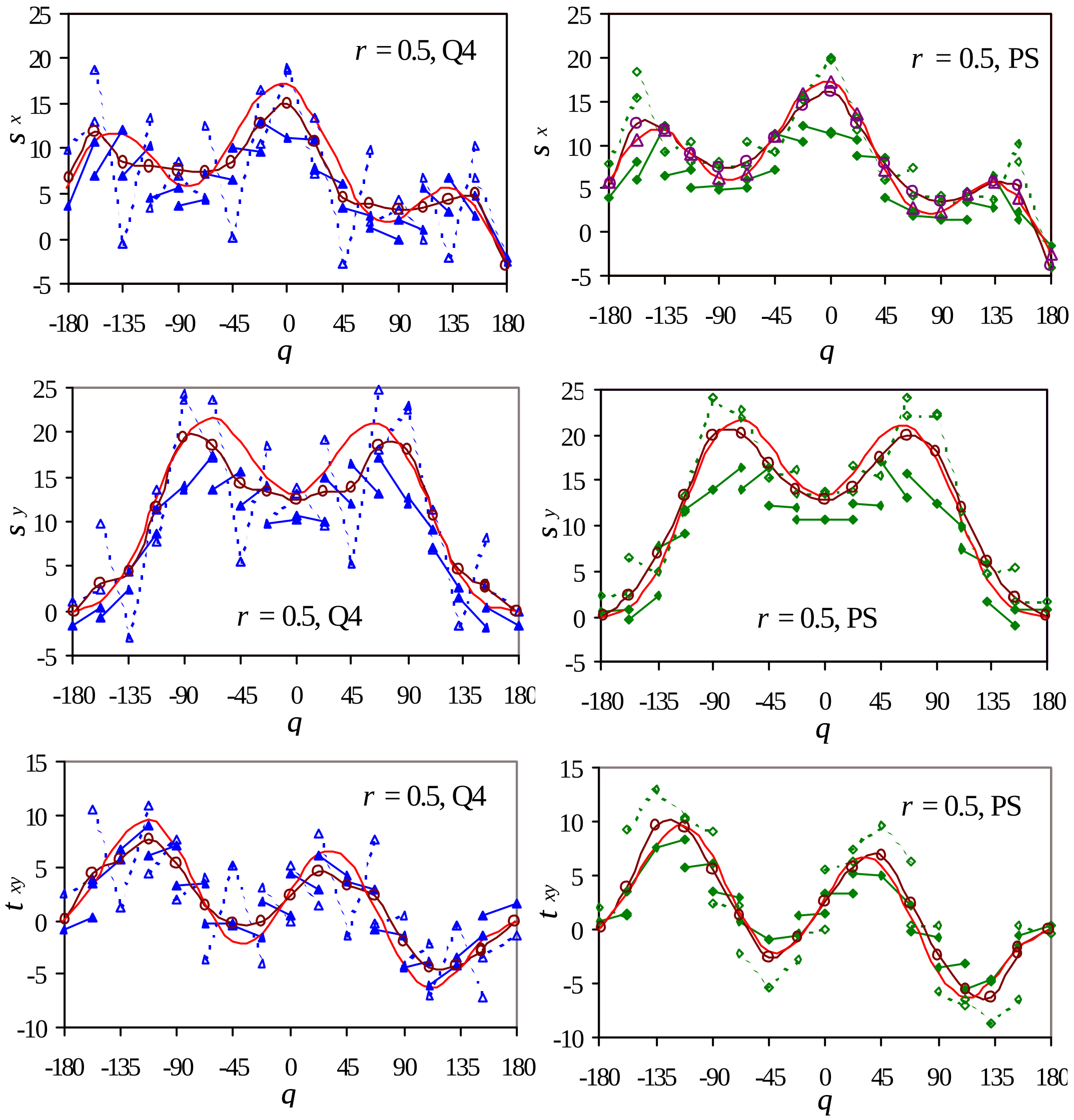


Fig. 2: Angular stress distributions along the circle r = 0.5.

As a general method, SAR can be used with traditional FEs as well as the recently introduced meshless methods. It seems most suitable for the extended FEM (Daux et al. [11]), where local displacement conditions are met by proper enrichment functions. With the use of SAR, the XFEM will not be dependent on the availability of actual crack tip fields and is thus likely to be powerful for complicated crack problems, e.g. cohesive cracks.

The stress field recovered from SAR has another advantage in that it gives an upper bound of the strain energy. If the displacement compatible element is used, the stress field obtained directly by the FEM would give a lower bound of the strain energy. Thus the strain energy will be bound from both sides. Hence, the value of the weighted L_2 norm in eqn (1) will give a natural estimate of the error of the FE results.

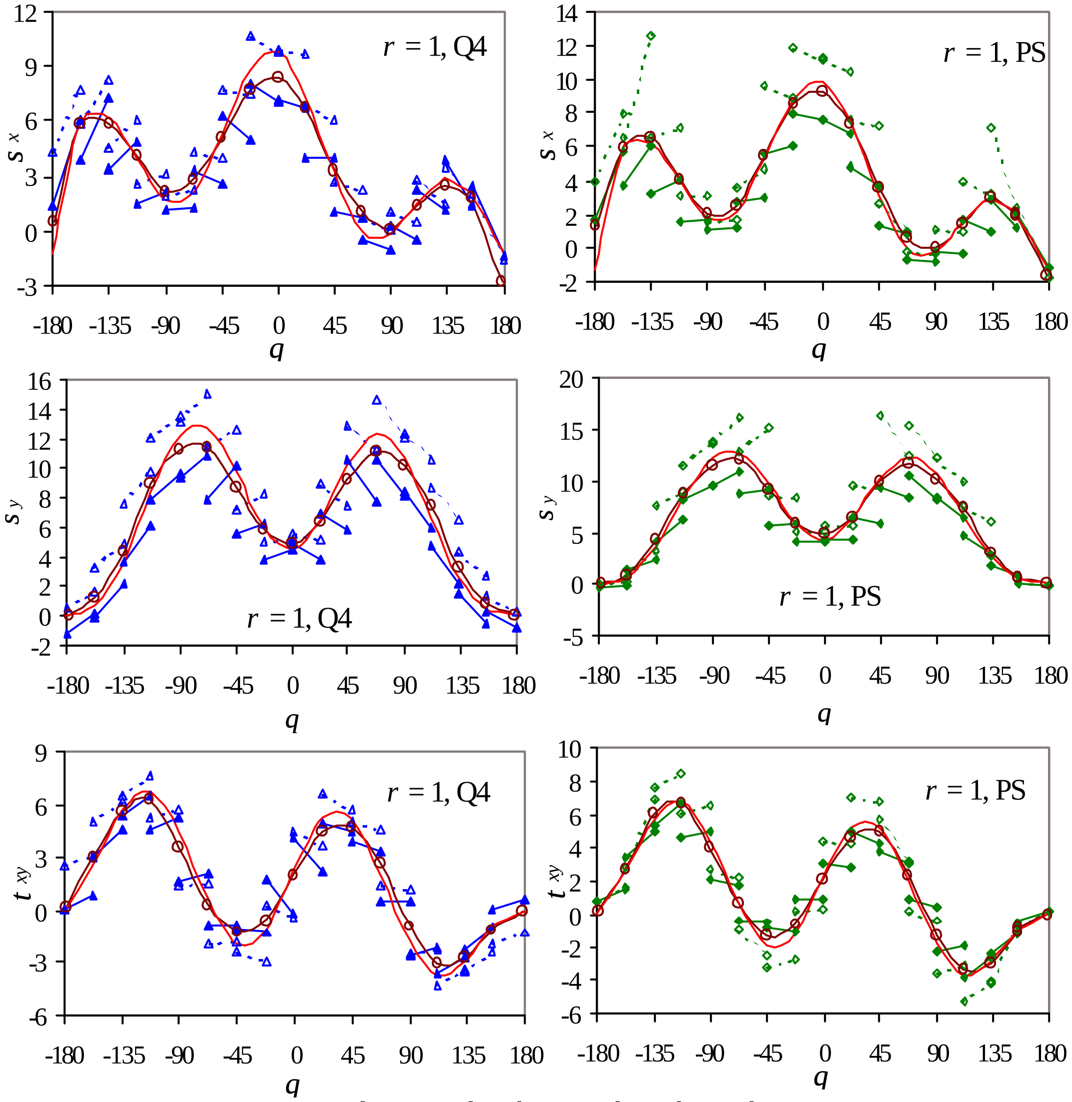
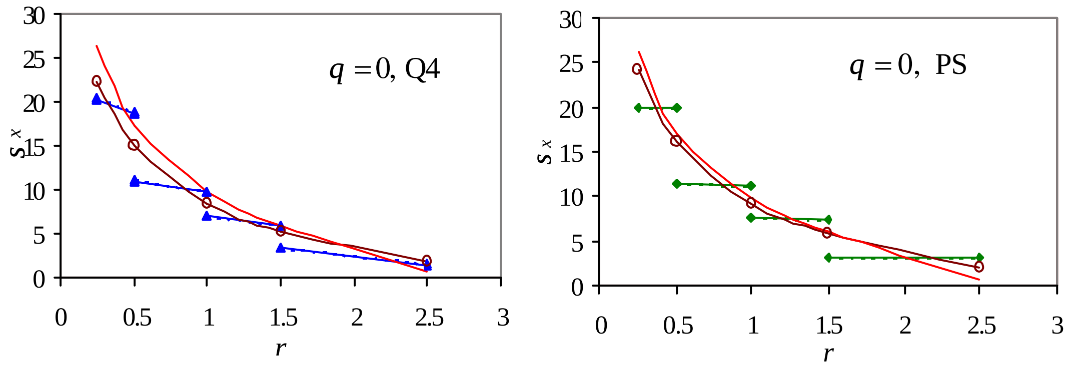


Fig. 3: Angular stress distributions along the circle r = 1.0



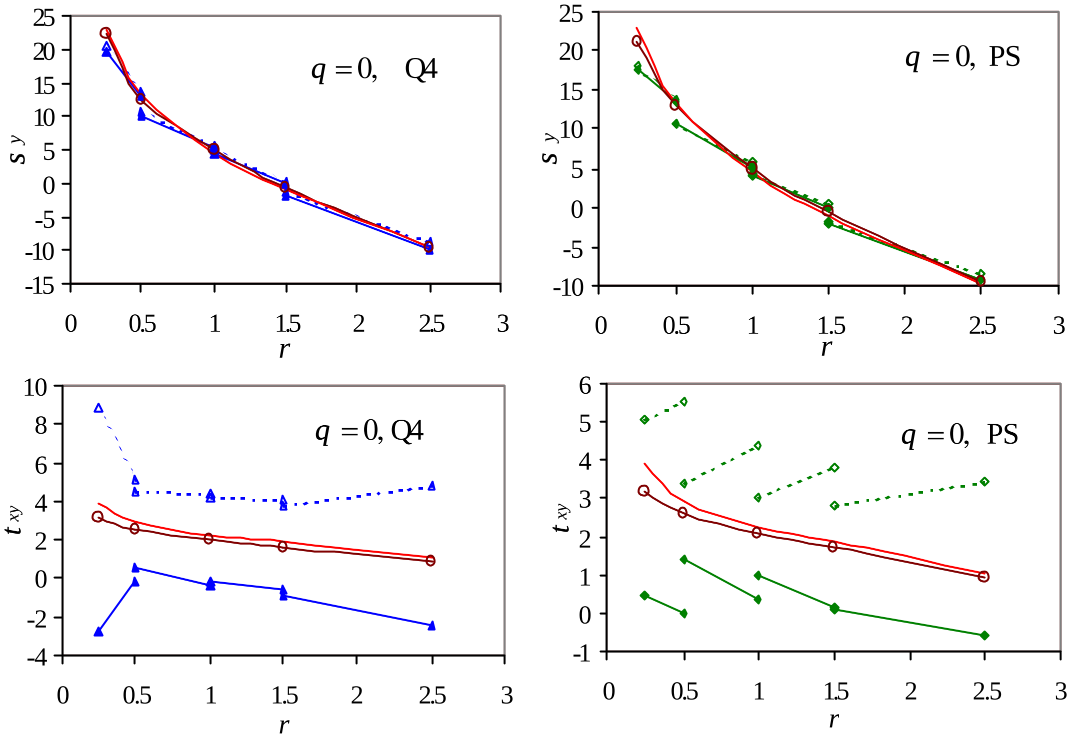


Fig. 4: Stresses along the line of extension of the crack.

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