A GENERAL WEIGHT FUNCTION FOR A SUBSURFACE CRACK IN A TWO DIMENSIONAL HALF-SPACE

M . Beghini[1], L. Bertini[1], V. Fontanari[2]
[1] Dip. di Ingegneria Meccanica, Nucleare e Produzione – University of Pisa (Italy)
[2] Dip. di Ingegneria dei Materiali e Tecnologie industriali - University of Trento (Italy)

ABSTRACT

The Weight Function (WF) for a subsurface crack parallel to the free surface of a two dimensional half-space is proposed and its general properties discussed. A matrix like formulation of the WF is considered to account for the loss of symmetry when the crack approaches the boundary of the half-space, giving rise to coupling effects between mode I and II of fracture. In order to study a completely general loading condition, the WF was built up by a symmetric and an anti-symmetric component. The 'multiple reference loading' approach was used to derive the analytical expression of the WF. To this purpose a parametric Finite Element (FE) analysis of the subsurface crack was set up, and the Stress Intensity Factors (SIFs) for several independent loading conditions were determined. The analysis was carried out for different positions of the crack with respect to the half-space boundary, and the dependence of the WF on this parameter was studied. An estimate of the accuracy is proposed and a practical application is finally presented in which the SIFs induced by a point like load travelling on the surface is evaluated.

1 INTRODUCTION

Several mechanisms of failure in mechanical components, such as spalling in rolling contact fatigue [1-2] or fatigue failure from subsurface defects in ultra high-cycle fatigue [3-4], can be explained by considering the initiation and propagation of subsurface cracks. In the scientific literature many different approaches to predict crack nucleation and propagation can be found [1-9]. If the distance of the crack from the body surface is sufficiently high compared to the crack length, the Griffith crack solution can be assumed as representative of the problem. This approximation turns to be questionable for subsurface cracks immediately below the component surface, as observed in many contact fatigue problems or for cracks that have been propagating over a length comparable with their distance from the surface. The FE have been used with success to evaluate the Stress Intensity Factors (SIF) under complex loading conditions [5] and also to predict the preferred crack propagation paths [7,8]. However, the FE method, since very powerful, is unfortunately a time-consuming technique particularly when many analyses have to be carried out to predict the crack propagation. Therefore, the development of more efficient approaches appears very useful. The WF method seems to be particularly efficient for this kind of problems. A WF has been recently proposed [9] to analyse a similar problem of a subsurface crack subjected to pure Mode II in rolling contact fatigue. However, on the authors knowledge, a completely general WF for subsurface cracks has been not yet presented in the literature.

The present paper is aimed at determining a general formulation of the WF, by which the Fracture Mechanics (FM) problem for a subsurface crack parallel to the external surface of a two-dimensional half space can be computed for every loading condition, including the typical contact loading. To this purpose, a matrix like WF is proposed to account for the coupling effect arising in non symmetrical problems, such as the subsurface crack in a semi-infinite body, when the distance from the free surface is comparable with the crack length. The WF has been built up into a symmetrical and anti-symmetrical component, as usual for embedded cracks, thus allowing for a straightforward evaluation of the FM parameters under a completely general loading condition. The 'direct assessment method' based on multiple reference loading cases [11] is adopted to determine the diagonal and off-diagonal terms of the WF. To this purpose a FE analysis was carried out for different loading conditions and for different ratios between the crack length and its distance from the half-space free surface. The dependence of the WF coefficients on this geometrical parameter was studied. The accuracy of the SIFs calculated by the WF is evaluated by comparison with equivalent values determined by the FE analysis. A practical application is finally presented in which the SIFs induced by a point like load travelling on the surface including the friction force, are evaluated.

2 MATHEMATICAL FORMULATION OF THE PROBLEM

An embedded crack in a semi-plane having length $2a$ and distance b from the semi-plane surface is represented in fig.1. In order to rigorously define the FM parameters (in particular regarding the sign of K_{II} of which a general definition is not usually adopted), a local Cartesian reference system is defined at each crack tip A and B. The two local reference systems share the y-axis while they have opposite x-axes (x_A and x_B repect.) each of them pointing toward the direction of crack propagation at the two tips. It is worth noting that the y-axis is the only axis of symmetry for the problem, its versus is univocally defined as it is chosen toward the nearest free surface. This is not the only possible choice, however, the adopted systems is useful for representing the axial symmetry of the problem, in which the quantities referred to the y-axis are the same for the two opposite parts of the crack. As discussed in [10], the relative displacements of the points C^+ (D^+) and C^- (D^-) located on the upper (y+) and lower (y-) crack edge respectively, define the sign of the SIFs when C (or D) approaches the tip A (or B). In particular: K_I for the tip A (B) is positive if the y component of the displacement of point C^+ (D^+) is higher than the correspondent component of point C^- (D^-). Similarly, K_{II} at the tip A (B) is positive if the x_A (x_B) displacement component of point C^+ (D^+) is higher than the corresponding component of the point C^- (D^-).

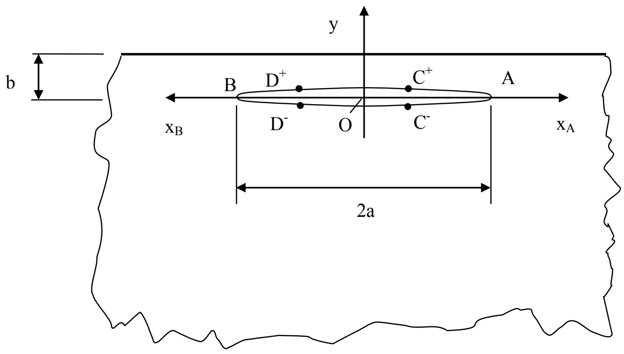

Fig.1 Subsurface crack parallel to the semiplane surface

As follows from the original formulations of Bueckner [12] and Rice [13], by considering the energy associated to a crack in a linear elastic fracture mechanics, the WF derives from the elastic solution for one reference load system. With reference to Figure 1 in the case of an embedded crack, when the parameter a/b ->0 (the crack behaving as a Griffith crack in an infinite body), the SIF can be evaluated for any loading condition at tips A and B respectively by eqns. 1a and 1b.

Since a general loading condition on the crack can be represented by an appropriate combination of symmetrical (S) and anti-symmetrical (A) stress distribution along the crack, the SIF can be obtained by superposing the two contribution with an appropriate sign, distinguishing between the right and the left crack tip:

$$K_M(a)^A = \int_0^a h^M{}_S(x,a)\cdot S_S(x)\cdot dx + \int_0^a h^M{}_A(x,a)\cdot S_A(x)\cdot dx \tag{1a}$$

$$K_M(a)^B = \int_0^a h^M{}_S(x,a)\cdot S_S(x)\cdot dx - \int_0^a h^M{}_A(x,a)\cdot S_A(x)\cdot dx \tag{1b}$$

M is either I or II in dependence on the mode of opening and S(x), the nominal stress produced along the crack line (either normal σ(x) or tangential τ(x)). Eqns. (1a) and (1b) hold for the local reference systems described in fig. 1. As a consequence two expressions of the WF have to be defined.

For a non symmetrical problem, one reference load system is not sufficient to define the WF as two SIF components are expected and the energy associated to the crack depends on both of them. As discussed in [10,11] a matrix like formulation of the WF is necessary. In the present case the geometrical parameter governing the problem is the ratio $r=a/b$ and therefore the following general expression holds, where respectively for the tip A and the tip B, the anti-symmetrical contribution has to be summed to (+), or subtracted from (-) the symmetrical contribution:

$$\begin{pmatrix} K_I(r) \\ K_{II}(r) \end{pmatrix}^{A/B} = \int_0^a \left[\begin{pmatrix} h^{I\sigma}{}_S(x,r) & h^{I\tau}{}_S(x,r) \\ h^{II\sigma}{}_S(x,r) & h^{II\tau}{}_S(x,r) \end{pmatrix} \cdot \begin{pmatrix} \sigma(x) \\ \tau(x) \end{pmatrix}_S \pm \begin{pmatrix} h^{I\sigma}{}_A(x,r) & h^{I\tau}{}_A(x,r) \\ h^{II\sigma}{}_A(x,r) & h^{II\tau}{}_A(x,r) \end{pmatrix} \cdot \begin{pmatrix} \sigma(x) \\ \tau(x) \end{pmatrix}_A \right] \cdot dx \tag{2}$$

The analytical expressions of the WF can be assumed, accounting for their general properties. In particular, as the r ratio approaches zero the crack can be assimilated to the Griffith crack in an infinite body for which the following uncoupled relationships hold:

$$\begin{aligned} r \to 0 \qquad & h^{I\sigma}{}_S(x,r) = h^{II\tau}{}_S(x,r) = \frac{2}{\sqrt{\pi\cdot a}}\cdot\sqrt{1-\left(\frac{x}{a}\right)^2} \\ & h^{I\sigma}{}_A(x,r) = h^{II\tau}{}_A(x,r) = \frac{2}{\sqrt{\pi\cdot a}}\frac{x}{a}\cdot\sqrt{1-\left(\frac{x}{a}\right)^2} \\ & h^{II\sigma}{}_S(x,r) = h^{I\tau}{}_S(x,r) = h^{II\sigma}{}_A(x,r) = h^{I\tau}{}_A(x,r) = 0 \end{aligned} \tag{3}$$

Therefore, for solving the integral equations (2) the following expressions were assumed for representing the general WF:

$$h_S^{M\mu}(x,r,a) = \frac{2}{\sqrt{\pi\cdot a}}\sum_{i=0}^{n} c_S^{M\mu}{}_i(r)\cdot\left[1-\left(\frac{x}{a}\right)^2\right]^{i-\frac{1}{2}} \tag{4a}$$

$$h_A^{M\mu}(x,r,a) = \frac{2}{\sqrt{\pi\cdot a}}\cdot\frac{x}{a}\cdot\sum_{i=0}^{n} c_A^{M\mu}{}_i(r)\cdot\left[1-\left(\frac{x}{a}\right)^2\right]^{i-\frac{1}{2}} \tag{4b}$$

In order to fulfil the general asymptotic conditions, expressed by (3), the first coefficients corresponding to i=0 for the diagonal and off-diagonal terms of the WF are respectively:

$$c_S^{M\mu}{}_i(r) = c_A^{M\mu}{}_i(r) = 1 \quad for \quad i=0 \qquad and \quad M\mu = I\sigma \, or \, II\tau \tag{5a}$$

$$c_S^{M\mu}{}_i(r) = c_A^{M\mu}{}_i(r) = 0 \quad for \quad i=0 \qquad and \quad M\mu = II\sigma \, or \, I\tau \tag{5b}$$

The other coefficients corresponding to $i>0$ depend on the ratio r and a polynomial expression was adopted to represent this dependence:

$$c^{M\mu}_{(S/A)_i}(r) = \sum_{j=1}^{m} (\alpha^{M\mu}_{S/A})_{i,j} \cdot (r)^{j+1} \tag{6}$$

3 EVALUATION OF THE WF

To evaluate the eight components of the WF, several reference loading conditions have to be considered and for each of these, the SIF have to be determined at different r ratios. The SIFs were obtained by the FE method. Since the number of terms in eqns. 4 was limited to three (n=2), being the first of these defined by the asymptotical conditions, the other two coefficients can be evaluated by carrying out the analysis for two independent loading conditions at different values of r. The loading cases of fig. 3 were considered, by applying appropriate stress distributions on the crack edges, thus allowing to determine respectively the symmetrical (loading cases of Fig. 3 I) the anti-symmetrical (loading cases of fig. 3 II) part of the WF as expressed by eqn.(2).

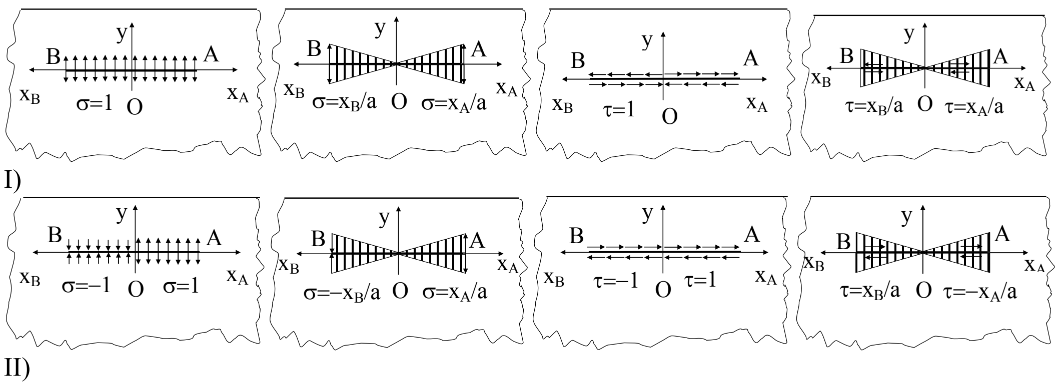

Fig. 3: loading conditions used in the analysis. I) symmetrical uniform and linearly variable stress, II) anti-symmetrical loading uniform and linearly variable stress

The results of FE analysis carried out for r varying in the range $0.05<r<3.1$ are reported in [14]. A reasonable compromise between the number of parameters in eqn. (6) and the accuracy of the WF was found by assuming m=3 in the series expansion. The corresponding $\alpha_{(M\mu)\, i,j}$ coefficients are reported in table I.

Symmetrical components				***Anti-symmetrical components***			
$\alpha_S{}^{I\sigma}{}_{i,j}$	*j=1*	*2*	*3*	$\alpha_A{}^{I\sigma}{}_{i,j}$	*j=1*	*2*	*3*
i=1	1.236	-0.482	0.069	i=1	0.350	0.058	-0.021
2	0.273	-0.050	0.005	2	0.118	0.090	-0.022
$\alpha_S{}^{II\sigma}{}_{i,j}$	*j=1*	*2*	*3*	$\alpha_A{}^{II\sigma}{}_{i,j}$	*j=1*	*2*	
i=1	-0.296	0.101	-0.015	i=1	0.298	-0.181	0.026
2	-0.140	-0.112	0.026	2	0.152	-0.211	0.036
$\alpha_s{}^{I\tau}{}_{i,j}$	*j=1*	*2*	*3*	$\alpha_A{}^{I\tau}{}_{i,j}$	*j=1*	*2*	
i=1	-0.271	0.188	-0.032	i=1	-0.330	0.101	-0.011
2	-0.183	-0.028	0.014	2	-0.099	0.104	-0.019
$\alpha_s{}^{II\tau}{}_{i,j}$	*j=1*	*2*	*3*	$\alpha_A{}^{II\tau}{}_{i,j}$	*j=1*	*2*	
i=1	0.158	-0.051	0.008	i=1	0.179	-0.021	-0.0004
2	0.152	-0.168	0.031	2	0.068	-0.013	-0.0037

Table I: coefficients $\alpha^{M\mu}{}_{i,j}$ for the WFs

The SIFs calculated by using the WF for the eight reference cases were compared with those obtained by the FE analysis. The relative differences between the WF and the FE SIFs for both Modes showed a satisfactory agreement being generally within 2.0% in the whole considered range of *r*. This results demonstrate the adequacy of the function (eqn. 4-6) for interpolating the FE results and they indicate that, for the considered range of *r*, an increment of terms is not necessary as the relative differences are within the estimated range of accuracy for the FE SIF evaluation.

3 AN APPLICATIVE EXAMPLE

With reference to figure 4, a plane body carrying a subsurface crack and loaded by a force uniformly distributed through thickness having intensity P (force per unit thickness) is considered. The force P is applied inward the body normally to the surface at a variable distance L from the crack mouth thus reproducing the conditions of a travelling force. Inertia forces are neglected. Material is considered linear elastic and no contact between crack edges is taken into account. Under these assumptions material overlapping is permitted even though without physical meaning. The nominal stress produced by P in the uncracked body to be used in eqn. (2) can be deduced by the analytical Boussinesq solution [15] and subdivided in a symmetrical and an anti-symmetrical component. By solving the integral equation in eqn. (2) the K_I and K_{II} values for different relative load position (L/a) and different *r* were calculated, and the obtained trends are reported in figure 5, together with results of a FE analysis of the problem. In this case the following characteristic SIF was adopted as normalising factor: $K_O = P \cdot \sqrt{\frac{\pi}{a}}$. A very good agreement between FE and WF results was found, being the relative difference almost within the 0.5%.

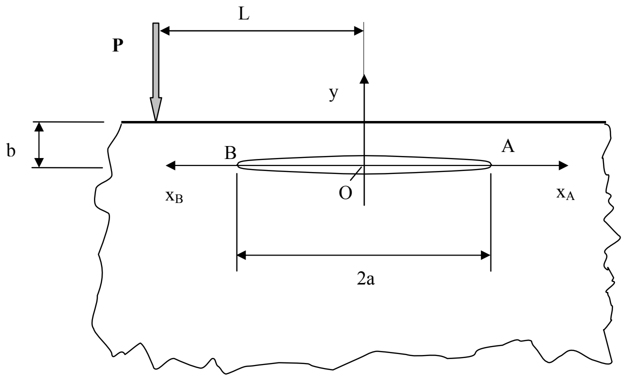

Fig. 4:travelling point like load on the surface

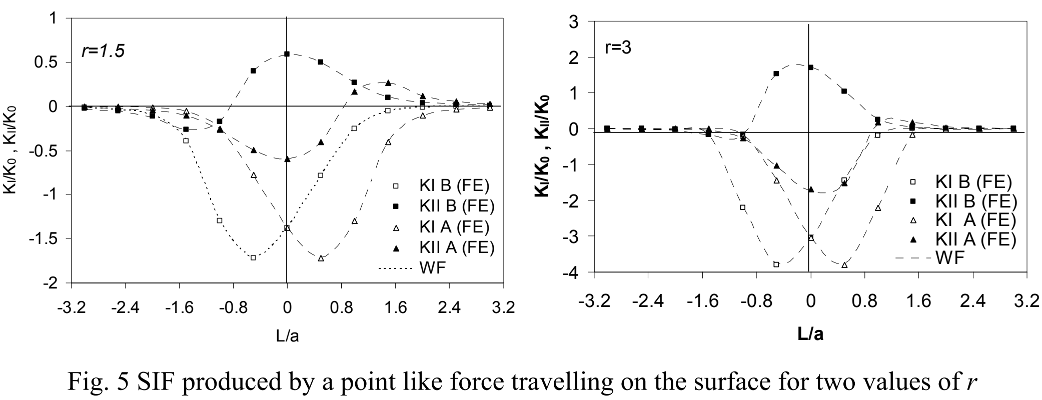

Fig. 5 SIF produced by a point like force travelling on the surface for two values of *r*

3 CONCLUSIONS

An analytical matrix like formulation of the WF for a general two-dimensional subsurface crack was proposed. A FE analysis was carried out to determine the SIF for several independent loading cases and the results were used for evaluating the numerical coefficients of the WF. The obtained WF reproduces the FE results with a good accuracy thus showing to be adequate for interpolating the FE results. An independent applicative example was considered to test the usefulness of the WF. The SIF produced by a point like force moving on the surface were calculated by WF and FE obtaining very small relative differences.

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