STRESS INTENSITY FACTOR OF 3D PLANE CRACKS UNDER MODE I LOADING

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ABSTRACT

In this paper, a numerical method for the determination of the mode I stress intensity factor of an arbitrary plane crack embedded in an infinite isotropic elastic body, is proposed.
This method is based on the three-dimensional weight-function theory of Bueckner-Rice, that gives the variation of the stress intensity factor along the crack front arising from some small arbitrary coplanar perturbation of the front. It is closely linked to previous works of Bower and Ortiz but much simpler in its numerical implementation.
The main advantage is that only one dimensional integrals along the crack front are involved so that only the one dimensional meshing of the crack front is needed, and not the 3D meshing of the whole body as in the finite-element method.
Applications include the asymptotic behavior of the stress intensity factor along the crack front near an angular point and the fatigue propagation path of mode I plane cracks undergoing a large number of loading cycles.

KEYWORDS

Linear elastic fracture mechanics, stress intensity factor, mode I crack, 3-D weight function, 3-D plane/flat crack, perturbation method, angular point, fatigue propagation path.

INTRODUCTION

Let us consider a plane crack with arbitrary contour $\mathcal{F}$, embedded in an infinite isotropic elastic body and loaded in pure mode I through some uniform stress σ_∞ applied at infinity (see fig. 1). The aim of this paper is to determine the mode I stress intensity factor (SIF) along $\mathcal{F}$. A classical method would be to use the finite element method (FEM), but here we propose an alternative method whose main advantage is to restrict the meshing operations to that of the front instead of the whole body. It is based on the three-dimensional weight-function theory derived by Gao and Rice : in [7] the half plane crack is studied, in [4] the penny shaped one and finally, in [8] the theory for any plane crack is presented. It was used by Bower and Ortiz [1, 2, 3] to study several problems concerning a half plane crack. However, the originality of our work lies in the simplification of the numerical implementation and in the applications studied : the asymptotic behavior of the SIF near an angular point of the front and some examples of fatigue propagation paths.

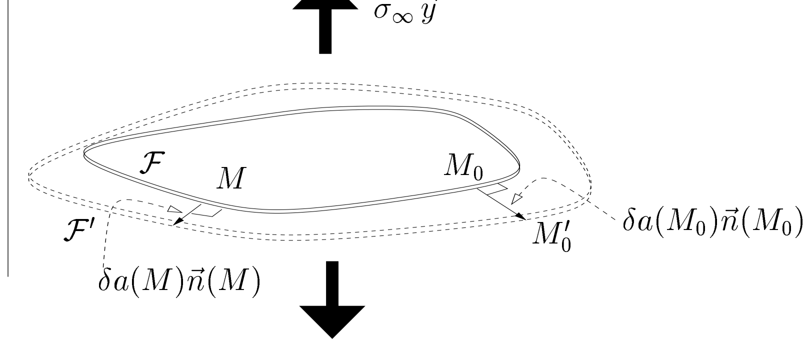


Figure 1: Arbitrary plane tensile crack in an infinite body under uniform stress σ_∞.

PRINCIPLE OF THE METHOD

Three-dimensional weight-function theory

As the crack advances, under constant loading, by a small distance $\delta a(M)$ in the direction perpendicular to the front $\mathcal{F}$ like in figure 1, Rice [8] has shown that to first order in δa, the SIF at point M_0' of the new front defined by

$$\overrightarrow{M_0 M_0'} = \delta a(M_0)\vec{n}(M_0), \tag{1}$$

can be approximated by $K(M_0) + \delta K(M_0)$ where

$$\delta K(M_0) = \frac{1}{2\pi} PV \int_{\mathcal{F}} \frac{W(M, M_0)}{D^2(M, M_0)} K(M)[\delta a(M) - \delta_* a(M)]\mathrm{d}M \tag{2}$$

$D(M, M_0)$ is the distance between the points M and M_0, $W(M, M_0)$ is a two-variable function linked to the weight function of the crack[1]. The function W along the new crack front can be, itself, updated by $W(M_0, M_1) = W(M_0, M_1) + \delta W(M_0, M_1)$ where the variation of W is given, also to first order in δa, by :

$$\delta W(M_0, M_1) = \frac{D^2(M_0, M_1)}{2\pi} PV \int_{\mathcal{F}} \frac{W(M, M_0)\, W(M, M_1)}{D^2(M, M_0)\, D^2(M, M_1)} [\delta a(M) - \delta_{**} a(M)]\mathrm{d}M \tag{3}$$

These formulae are legitimate for special normal advances $\delta_* a(M)$ and $\delta_{**} a(M)$ that preserve the shape of the front and such that $\delta_* a(M_0) = \delta a(M_0)$, $\delta_{**} a(M_0) = \delta a(M_0)$ and $\delta_{**} a(M_1) = \delta a(M_1)$ so as to ensure the existence of the Principal Value (PV) integrals. One can always define some combination of translatory motion, rotation and scaling that verifies all these conditions.

As the quantities in right-hand side of Eqn. 2 and 3 concern only the front $\mathcal{F}$, this theory allows to calculate the SIF and function W along the perturbed one $\mathcal{F}'$ if the SIF and function W are known for the initial one $\mathcal{F}$.

Determination of the SIF and of the function W

Assume now that the functions $K(M)$ and $W(M, M_0)$ are known for one crack shape $\mathcal{C}$ and that a succession of very close to each other, intermediate cracks $\mathcal{F}_k$, $k = 0 \ldots n$, such that $\mathcal{F}_0 = \mathcal{C}$ and $\mathcal{F}_n = \mathcal{F}$, can be constructed. Then by applying Eqn. 2 and 3 successively between $\mathcal{F}_0$ and $\mathcal{F}_1$, between $\mathcal{F}_1$ and $\mathcal{F}_2$, ... and finally between $\mathcal{F}_{n-1}$ and $\mathcal{F}_n$ the SIF $K = K + \delta K$ and the function $W = W + \delta W$ along $\mathcal{F}_1$,$\mathcal{F}_2$, ... and finally $\mathcal{F}_n = \mathcal{F}$ can be obtained.

[1] More exactly to the SIF at the point M of $\mathcal{F}$ induced by unit point forces exerted on the point M' of the crack lips in the direction $\pm\vec{y}$, when M' approaches M_0 (see Rice [8] for the exact definition).

In the sequel, we restrict our attention to a bounded plane crack $\mathcal{F}$ that can be derived from a penny shaped one $\mathcal{C}$ of center O and radius R for which $K(M) = 2\sigma_\infty\sqrt{R/\pi}$ and $W(M, M_0) = 1$ (see for instance, Rice [8]), but each crack shape for which the functions $K(M)$ and $W(M, M_0)$ are known could be chosen as starting point (see [1, 2, 3] for the half plane crack).

Meshing

The initial crack front $\mathcal{F}$ is meshed with N points P_i, $i = 0 \dots N-1$. The N nodes P_i^0 of the reference front $\mathcal{C}$ are constructed through intersection of $\mathcal{C}$ with the lines (OP_i). The segments $[P_i^0 P_i]$ are then cut into n pieces to create $n-1$ intermediate meshes $\mathcal{F}_k$, $k = 1, n-1$ with nodes $P^k_{j,j=0,N-1}$. As the

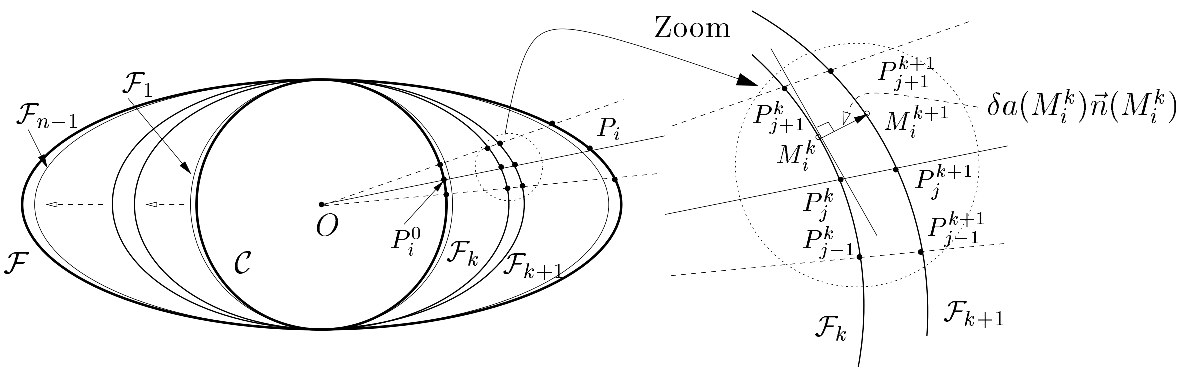


Figure 2: Intermediate cracks between the reference front $\mathcal{C}$ and the final one $\mathcal{F}$.

vector $\overrightarrow{P_j^k P_j^{k+1}}$ is not in general normal to the front $\mathcal{F}_k$ (see fig. 2), the Eqn. 2 and 3 don't give the SIF at point P_j^{k+1} as a function of its values on nodes $P^k_{i,i=0,N-1}$ of $\mathcal{F}_k$. Therefore a second set of meshes $M^k_{i,i=0,N-1}$ of $\mathcal{F}_k$, $k = 0, n$ is constructed by projection of the nodes M_i^k of $\mathcal{F}_k$ onto the arc of a circle passing through the 3 successive nodes P^{k+1}_{j-1}, P^{k+1}_j,P^{k+1}_{j+1}, of $\mathcal{F}_{k+1}$ with, as initialization, $M_i^0 = P_i^0$, $i = 0, N-1$. The stress intensity factor and function W are then computed on the nodes $M^k_{i,i=0,N-1}$, $k = 1, n$ of this set of meshes. If the meshes become too distorted, remeshing is done.

Calculation of the integrals involved

To calculate each $\delta K(M_i^k)$, the PV part, around M_i^k, is extracted from Eqn. 2 and rewritten in the form (by taking into account the fact that $D(M, M_i^k) \sim |s(M) - s(M_i^k)|$ in the neighborhood of M_i^k) :

$$PV \int_{[M^k_{i-1} M^k_{i+1}]} a \frac{(s(M) - s(M_i^k))(s(M) - b)}{(s(M) - s(M_i^k))^2} \mathrm{d}s(M) =$$
$$a\left[s(M^k_{i+1}) - s(M^k_{i-1}) + (s(M_i^k) - b) \ln \frac{s(M^k_{i+1}) - s(M_i^k)}{s(M_i^k) - s(M^k_{i-1})} \right] \tag{4}$$

where a, b are interpolation constants, $s(M)$ some curvilinear abscissa along the crack front. The integral over the rest of the front is regular and calculated by quadratic interpolation over each interval.
To calculate $\delta W(M_i^k, M_j^k)$, a similar procedure is employed. Nevertheless, attention must be paid to the fact that the PV concerns both points M_i^k and M_j^k.
One should notice that the procedure is less complicated than the one used by Bower and Ortiz [1], but gives comparable results, as shown below.

The elliptical crack

For an elliptic crack with major axis b and minor axis a subjected to some uniform tensile loading σ_∞, Irwin [6] has shown that :

$$K(M) = \frac{\sigma_\infty\sqrt{\pi a}}{\mathrm{E}(k)}\left(\frac{\sin^2(\theta)+\alpha^4\cos^2(\theta)}{\sin^2(\theta)+\alpha^2\cos^2(\theta)}\right)^{1/4} \quad \text{with} \quad \mathrm{E}(k) = \int_0^{\pi/2}(1-k^2\sin^2 x)^{1/2}\mathrm{d}x \tag{5}$$

where θ is the polar angle of M, $\alpha = a/b$, $k = \sqrt{1-\alpha^2}$ and $\mathrm{E}(k)$ denotes the elliptic integral of the second kind. The numerical results obtained for different values of α are in good agreement with this analytical result.

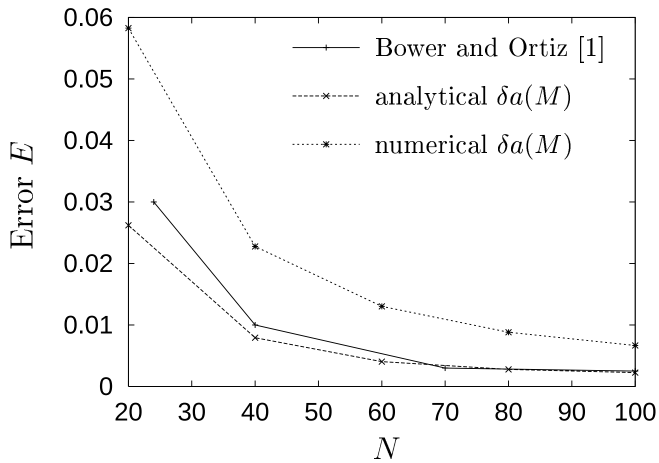


Figure 3: Error E obtained by different methods.

To compare our results to those of Bower and Ortiz [1], the case $\alpha = 1/3$ is considered. Figure 3 shows the value of the error E, defined by :

$$E = \sqrt{\frac{1}{\text{perimeter}}\int_{\mathcal{F}}\frac{(K_{num}(M)-K_{exact}(M))^2}{K_0^2}\mathrm{d}M} \tag{6}$$

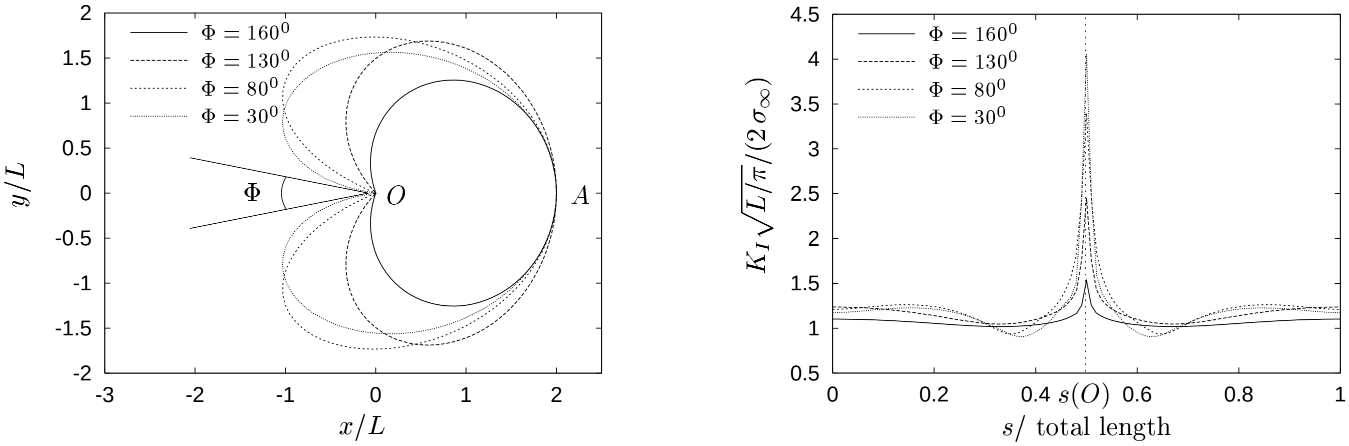
as a function of the number of nodes N, a constant maximum step size of $0.005a$ between the intermediate fronts $\mathcal{F}_k$ and $\mathcal{F}_{k+1}$ being used. K_0 denotes the uniform SIF along the initial penny-shaped crack $\mathcal{C}$ of radius a. When the advance of the front is given analytically like in the work of Bower and Ortiz, E is of the same order although our procedure of integration is simpler. However, when the advance is computed numerically, the error is obviously slightly increased, but reasonable enough to allow us to study cracks with more complex shapes.

Asymptotic behavior of SIF near an angular point of the front

The stress intensity factor along the front of several "heart shaped cracks", like the ones depicted in figure 4(a), with different opening angles Φ is given in figure 4(b).
Leblond and Leguillon [5] have shown that near the angular point O of the front, the SIF behaves in the following manner :

$$K(M) \propto |s(M)-s(O)|^{1/2+\alpha} \text{ when } M \to O, \tag{7}$$

where α depends only on the opening angle Φ and verifies $\alpha < -1/2$ so that the SIF becomes infinite at the notch point. The peak in figure 4(b) is the numerical manifestation of this propriety.
The scalar α can be computed by fitting the behavior 7 with the results, around the corner point O, of figure 4(b). The values obtained are given in figure 5 for several angles Φ. Errors are due to the dependence of the results upon the points chosen for fitting and to the numerical errors in the computation of the SIF. They are all the greater as the shape of the crack is more different from the



(a) Shape

(b) SIF

Figure 4: Heart shaped cracks

initial circle $\mathcal{C}$ i.e. as Φ is smaller. Nevertheless, our values are relatively close to the ones obtained by Leblond and Leguillon [5] by a more precise method in spite of the uncertainties linked to our method.

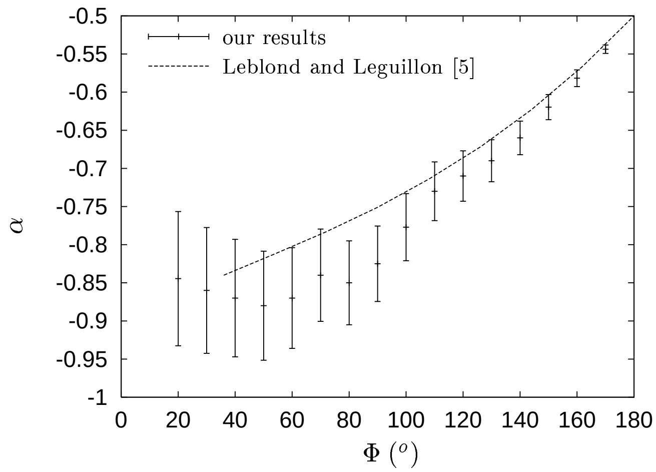


Figure 5: Computed values of the exponent α

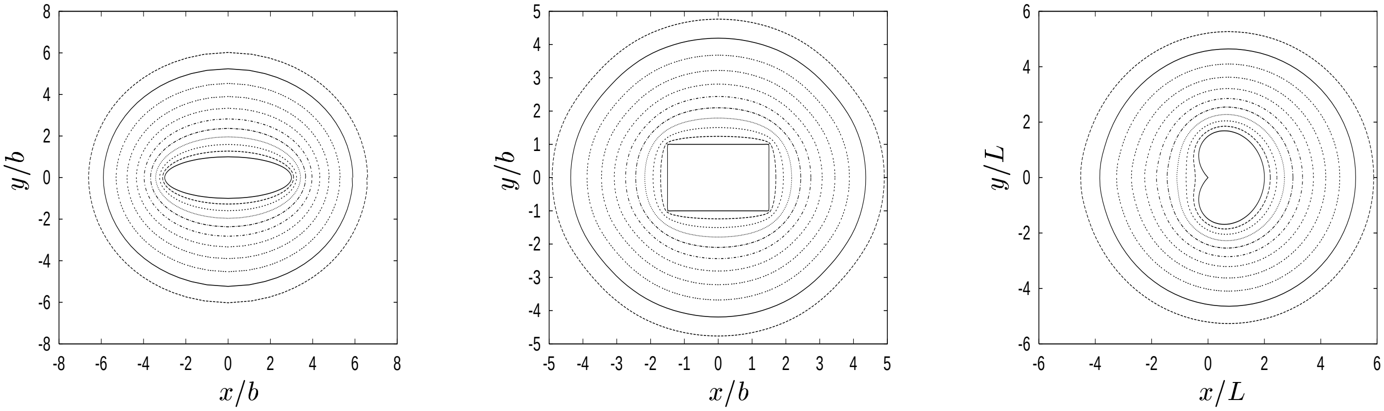
SOME EXAMPLES OF FATIGUE PROPAGATION PATHS

Let us now consider a bounded plane crack $\mathcal{F}$ loaded in pure mode I through some uniform cyclic tensile stress applied at infinity. Suppose that the crack propagation rate is given by Paris' law and that during a few, say n_c cycles, the variation of mode I intensity factor $\Delta K(M)$ along the front remains constant so that :

$$\frac{\mathrm{d}a(M)}{\mathrm{d}n} = C(\Delta K(M))^\beta \quad \Longrightarrow \quad \delta a(M) = n_c.C.(\Delta K(M))^\beta \tag{8}$$

where C and β are material constants and $\delta a(M)$ the normal crack advance after n_c cycles at the point M of the front.
Once the SIF and W are computed for the initial front $\mathcal{F}$ as explained above, the propagation path of this flaw can be determined by

1. applying Eqn. 8 to determine the displacement $\delta a(M)$ of the front;
2. using Eqn. 2 and 3 to update the SIF and the function W, and Eqn. 1 to obtain the new front;
3. repeat, as many times as required, the two preceding operations.



(a) Ellipse (b) Rectangle (c) “Heart”

Figure 6: Propagation of some cracks for ($\sigma_\infty = 1$ Pa, $n_c C = 0.005$ Pa^{-2}, $\beta = 2$).

Figure 6 shows, as typical examples, positions of the front during the propagation of initially elliptic, rectangular and heart shaped cracks each $20n_c$ cycles of loading. It appears that the crack becomes and remains circular after a certain time. This seems to be a general feature of the fatigue propagation of *bounded* mode I cracks embedded in an infinite body.

CONCLUSION

Since only the meshing of the initial front is needed, the procedure depicted above is an efficient tool for solving problems concerning a flat crack subjected to mode I loading. For instance, we have seen that the asymptotic behavior of the SIF near an angular point of the front and the fatigue propagation path of bounded cracks over a large distance can easily be computed.
Nevertheless, it would be interesting to extend the method to finite bodies to broaden the field of applications. But we do not know yet how to take into account boundary effects.

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