STRESS INTENSITY FACTOR ERROR INDEX FOR FINITE ELEMENT ANALYSIS WITH SINGULAR ELEMENTS

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ABSTRACT

An error index for the stress intensity factor (SIF) obtained from the finite element analysis (FEA) results using singular elements is proposed. The index was developed by considering the facts that the analytical function shape of the crack tip displacement is known and that the SIF can be evaluated from the displacements only. The advantage of the index is that it has the dimension of the SIF and converges to zero when the actual error of the SIF by displacement correlation technique converges to zero. Numerical examples for some typical crack problems, including a mixed mode crack, whose analytical solutions are known, indicated the validity of the index. The degree of actual SIF error seems to be approximated by the value of the proposed index.

KEYWORDS

Fracture Mechanics, Stress Intensity Factor, Finite Element Method, Error Index, Singular Element

INTRODUCTION

It is popular to evaluate the integrity of a cracked structure under arbitrary loads by comparing the stress intensity factor (SIF) for the crack with the critical value peculiar to the material. The SIF is often evaluated from finite element analysis (FEA) results and it is effective particularly when the SIF solution for the crack under specific load condition is not known, while the error estimation of the obtained SIF is very important.

In the past, many techniques have been proposed for FEA of a cracked structure in order to express and evaluate the stress singularity of the stress at the crack tip. Among these, one of the most popular techniques is to apply singular element (SE), which Barsoum [1] and Henshell and Shaw [2] proposed independently, to realize the crack tip stress singularity. In this case, the SIF is usually evaluated by Tracey's formula [3] (Displacement Correlation Technique, hereafter referred to as DCT). The feature of this technique is that a SIF of practical accuracy can be obtained by comparatively coarse mesh division. So, many researchers have been trying to answer the question "how coarse the SE can be to secure the SIF accuracy?" However, the load conditions as well as the SE size has become known to affect the SIF accuracy. Thus, it is generally accepted that an optimum SE size that satisfies arbitrary conditions does not exist [4].

Generally the accuracy of the SIF solution by FEA is improved by increasing the number of elements. However, since it is an engineering problem (and especially to take advantages of SE), it is desirable to obtain sufficiently accurate SIF by a mesh division as coarse as possible. This will be possible if we can estimate the error of the SIF obtained from one trial analysis. We can make corrections or judge whether the obtained SIF solution is applicable from a practical viewpoint. Fuenmayor et al. [5] applied the error index (expressed through the energy norm) which Zienkiewicz and Zhu [6] proposed for estimating errors in FEA results. However, since the error index is not expressed in terms of the SIF, one can only expect that the SIF error will be small when the index becomes small. We cannot know the degree of the actual SIF error. So we developed a new SIF error index that has the dimension of the SIF, based on the following three facts: (i) The analytical function forms of the crack tip displacements are known. (ii) Though incomplete, displacements on a SE represent a part of the analytical displacement distribution. (iii) The SIF can be evaluated from the displacements of crack tip elements.

In the following, we will first explain the concept of the error index which we have developed, and then demonstrate its validity by comparing our error index with the actual error for two typical crack problems whose analytical solutions are known.

PROPOSAL OF DCE (DISPLACEMENT CORRELATION ERROR) INDEX

Consider a polar-coordinate system (r, θ) as shown in Figure 1 where the crack tip is chosen as the origin and the crack surfaces as $\theta = \pm\pi$. In this case, the relative displacements $u^*(r, \theta)$ and $v^*(r, \theta)$ in the x and y directions between two symmetric points across the x axis can be related to only mode I and mode II deformations [7], respectively, and, by applying the asymptotic solutions of the displacements $u(r, \theta)$ and $v(r, \theta)$ in the x and y directions, are given as

$$\begin{bmatrix} u^*(r,\theta) \\ v^*(r,\theta) \end{bmatrix} \equiv \begin{bmatrix} u(r,\theta) - u(r,-\theta) \\ v(r,\theta) - v(r,-\theta) \end{bmatrix} = \sum_{n=1}^{\infty} \frac{r^{\frac{n}{2}}}{G} \begin{bmatrix} -A_{\mathrm{II}\,n} f_{\mathrm{II}u\,n}(\theta) \\ A_{\mathrm{I}\,n} f_{\mathrm{I}v\,n}(\theta) \end{bmatrix} \qquad (1)$$

where G is the shear modulus and the functions $f_{\mathrm{I}\,v\,n}$ and $f_{\mathrm{II}\,u\,n}$ are defined as follows:

$$\begin{bmatrix} f_{\mathrm{I}v\,n}(\theta) \\ f_{\mathrm{II}u\,n}(\theta) \end{bmatrix} = \begin{bmatrix} \frac{n}{2}\sin\left(\frac{n}{2}-2\right)\theta - \left(-\kappa + \frac{n}{2} + (-1)^n\right)\sin\frac{n\theta}{2} \\ -\frac{n}{2}\sin\left(\frac{n}{2}-2\right)\theta + \left(\kappa + \frac{n}{2} - (-1)^n\right)\sin\frac{n\theta}{2} \end{bmatrix} \qquad (2)$$

Here, suffixes I and II indicate the corresponding crack opening modes, and suffixes u and v represent the quantities corresponding to the displacements u and v, respectively. κ is $(3-4\nu)$ for plane strain or $(3-\nu)/(1+\nu)$ for plane stress when ν is Poisson's ratio. Note that the SIFs are given as $K_{\mathrm{I}} = \sqrt{2\pi}\ A_{\mathrm{I}\,1}$ and $K_{\mathrm{II}} = -\sqrt{2\pi}\ A_{\mathrm{II}\,1}$.

On the other hand, when $U(r, \theta)$ and $V(r, \theta)$ are the displacements of SEs from FEA, corresponding relative displacements $U^*(r, \theta) \equiv U(r, \theta) - U(r, -\theta)$ and $V^*(r, \theta) \equiv V(r, \theta) - V(r, -\theta)$ related to mode I and mode II deformations, respectively, are given by

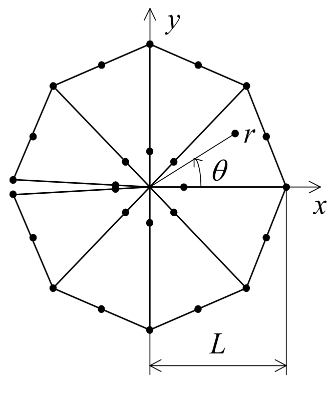

Figure 1: Singular crack tip elements

$$\begin{bmatrix} U^*(r,\theta) \\ V^*(r,\theta) \end{bmatrix} = \begin{bmatrix} 4U^*(L/4,\theta) - U^*(L,\theta) \\ 4V^*(L/4,\theta) - V^*(L,\theta) \end{bmatrix} \sqrt{\frac{r}{L}} + \begin{bmatrix} -4U^*(L/4,\theta) + 2U^*(L,\theta) \\ -4V^*(L/4,\theta) + 2V^*(L,\theta) \end{bmatrix} \frac{r}{L} \tag{3}$$

The Tracey's formula [3] is frequently used to evaluate the SIF from FEA results. That is, the SIF K_{DCT} is evaluated by letting Eqn. (3) correspond to the first two terms of Eqn. (1) on the crack surfaces ($\theta = \pi$) and it is given concretely, considering $f_{I\ v\ 1}(\pi) = f_{II\ u\ 1}(\pi) = \kappa + 1$, $A_{I\ 1} = K_{I\ DCT}/(2\pi)^{1/2}$, $A_{II\ 1} = -K_{II\ DCT}/(2\pi)^{1/2}$ and setting $G' \equiv (2\pi/L)^{1/2} G/(1+\kappa)$, as

$$\begin{bmatrix} K_{II\ DCT} \\ K_{I\ DCT} \end{bmatrix} = G' \begin{bmatrix} 4U^*(L/4,\pi) - U^*(L,\pi) \\ 4V^*(L/4,\pi) - V^*(L,\pi) \end{bmatrix} \tag{4}$$

Note that the K_{DCT} in Eqn. (4) is evaluated for $\theta = \pi$. It generally differs from the SIF evaluated in a similar way for other $\theta (\neq \pi)$, because SE displacements are not guaranteed to satisfy the angular characteristics which analytical expressions may show.

When we think of a sufficiently small region around a crack tip, terms higher than $O(r^{3/2})$ can be neglected in Eqn. (1). The relative displacements can be accurately expressed by the first two terms of Eqn. (1). If the true SIFs K_I and K_{II} are known, Eqn. (1) can be deduced for the crack surfaces, with $f_{I\ v\ 2}(\pi) = 0$ and $f_{II\ u\ 2}(\pi) = 0$ as follows.

$$\begin{bmatrix} u^*(r,\pi) \\ v^*(r,\pi) \end{bmatrix} = \frac{1}{G'} \sqrt{\frac{2\pi r}{L}} \begin{bmatrix} -A_{II\,1} \\ A_{I1} \end{bmatrix} = \frac{1}{G'} \sqrt{\frac{r}{L}} \begin{bmatrix} K_{II} \\ K_{I} \end{bmatrix} \tag{5}$$

On the other hand, the corresponding expressions $U^*(r, \pi)$ and $V^*(r, \pi)$ for the SEs are deduced, by substituting the SIFs $K_{I\ DCT}$ and $K_{II\ DCT}$ in Eqn. (4), as

$$\begin{bmatrix} U^*(r,\pi) \\ V^*(r,\pi) \end{bmatrix} = \frac{1}{G'} \sqrt{\frac{r}{L}} \begin{bmatrix} K_{II\,DCT} \\ K_{I\,DCT} \end{bmatrix} + \frac{2r}{L} \begin{bmatrix} U^*(L,\pi) - 2U^*(L/4,\pi) \\ V^*(L,\pi) - 2V^*(L/4,\pi) \end{bmatrix} \tag{6}$$

The nodal displacements in FEA are obtained by determining the unknown coefficients in the adopted displacement function through potential energy minimization process and Eqn. (3) does not necessarily coincide with Eqn. (1). Thus, the coefficients of r/L in Eqn. (6) are not zero unless the adopted displacement function can express the true displacement solution. However, since the SEs under consideration are conformal elements [1], FEA displacements tend to the exact solutions when the size of the elements approaches zero (note that the element size has to be decreased not only in the r direction but also in the θ direction). Then, the second term in Eqn. (6) converges to zero and K_{DCT} to the true value. This suggests the possibility of the second term in Eqn. (6) to become a SIF error index. We will now multiply the coefficient for r/L in Eqn. (6) with ($-G'/2$) and name it DCE index (Displacement Correlation Error Index) ΔK_{DCE}, which now has the dimension of a SIF.

$$\begin{bmatrix} \Delta K_{I\,DCE} \\ \Delta K_{II\,DCE} \end{bmatrix} = G' \begin{bmatrix} 2V^*(L/4,\pi) - V^*(L,\pi) \\ 2U^*(L/4,\pi) - U^*(L,\pi) \end{bmatrix} \tag{7}$$

Strong points of the DCE index, the proposed error index, are that (i) it can be directly calculated from the nodal displacements on the SEs, (ii) it converges to zero when the size of the SEs approaches zero and (iii) it has the dimension of a SIF. Thus, the DCE index differs from conventional error indexes, which generally focus on the convergence during iterative mesh refinements. The DCE index may therefore give a SIF error estimate from a single FEA results. This suggests the possibility of dramatically reducing efforts and costs in SIF analysis.

NUMERICAL EXAMPLES

In this section, FEA for two typical crack problems, whose analytical SIF solutions K_{ref} are known, were conducted by using SEs. Here the K_{DCT} is a SIF computed with DCT (Eqn. (4)) and ΔK_{DCE} is a DCE Index evaluated from the FEA results. Finally, K_{error}= ($K_{DCT} - K_{ref}$) was compared with the corresponding ΔK_{DCE}. In all cases, shearing modulus G of 79 GPa and Poisson's ratio ν of 0.3 were used. The number m of SEs investigated was 8 [8], 16, 24 and 30. For each m, a normalized SE size L/a of 1/3, 1/6, 1/12 and 1/24 was considered, approximately corresponding to the guideline proposed in the early days [8] (only the SEs were re-divided).

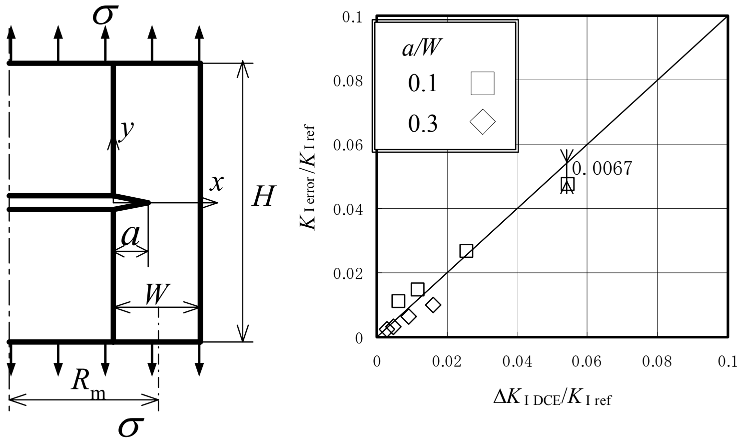

Figure 2: Actual SIF error $K_{I\ error}$ and DCE Index $\Delta K_{I\ DCE}$ (R_m/W= 9.5, H/W = 16, m = 16, ν = 0.3)

Circumferential Crack in a Cylinder under Uniform Tension

A circumferential crack in a cylinder under remote uniform stress σ= 9.8 MPa as in the left of Figure 2 was considered first. The dimensions of the cylinder were R_m = 95 mm in mean radius, W=10 mm thick and H = $16W$ = 160 mm long. The crack length was a = 1 or 3 mm. $K_{I\ error}$ was obtained by using Nied's analytical solution (a/W, $K_{I\ ref}$ MPam$^{1/2}$) = (0.1, 0.636), (0.3, 1.324) [9]. It was compared with $\Delta K_{I\ DCE}$ in the right of Figure 2 (m = 16).

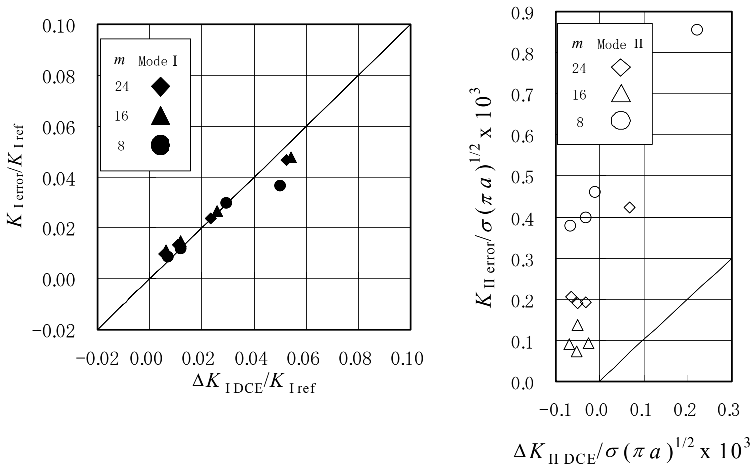

Figure 3: Effect of SE number m on actual SIF error K_{error} and DCE Index ΔK_{DCE} (for a/W = 0.1 in Figure 2)

There are four results for each a/W in Figure 2 and each result corresponds to the normalized SE size L/a, which has a positive correlation with the $\Delta K_{I\ DCE}$. The maximum difference between $K_{I\ error}$ and $\Delta K_{I\ DCE}$ in the figure is seen to be 0.67% of $K_{I\ ref}$ for the mark corresponding to a/W = 0.1 and L/a = 1/3. The figure shows the tendencies of $\Delta K_{I\ DCE}$ and $K_{I\ error}$ to decrease while L/a is made small. In addition, $\Delta K_{I\ DCE}$ and $K_{I\ error}$ are not very different for this problem.

Here as in the right of Figure 2, m = 16 was chosen without special notification. The effects of m on mode I and II K_{DCT}s for the case of a/W = 0.1 are summarized in Figure 3 left and right, respectively. Figure 3 left explains why we selected this specific m. There are four data for each three marks in Figure 3.

In the problem here, the analytical SIF for mode II is zero, so that $K_{II\ error}$= $K_{II\ DCT}$. Therefore $K_{II\ ref}$ = $\sigma(\pi a)^{1/2}$ was used to normalize $K_{II\ error}$ and $\Delta K_{II\ DCE}$ in Figure 3 right. We see from Figure 3 right that an increase in m does not necessarily contribute to the decrease in $K_{II\ error}$. It seems that this is due to the fact that the mode II SIF is zero for this problem, and that the tendency expected for conformal elements does not appear for small K_{II}, unless m and L are decreased together smoothly. On the other hand, we see from Figure 3 left that if we choose m to be 16 or more, the vertical distance between a mark and a line of unit slope crossing the origin (the difference between $K_{I\ error}$ and $\Delta K_{I\ DCE}$) becomes approximately constant. Thus, the effect of m on the mode I SIF can be disregarded.

Center Slant Cracked Rectangular Plate Subjected to Uniform Tension

The problem of a center slant cracked rectangular plate under uniform stress $\sigma = 9.8$ MPa in Figure 4 left was considered. The dimensions of the plate were $2W = 30$ mm wide and $2H = 60$ mm high. The crack had a length of $2a = 6$ or 12 mm and an angle of α=30°. K_{error} was obtained by using Kitagawa's analytical solution ($K_{\mathrm{I\,ref}}$, $K_{\mathrm{II\,ref}}$) = (0.735, 0.415), (1.138, 0.605) MPam$^{1/2}$ [10]. It was compared with $\Delta K_{\mathrm{I\,DCE}}$ in the right of Figure 4. This figure is the result for $m = 24$.

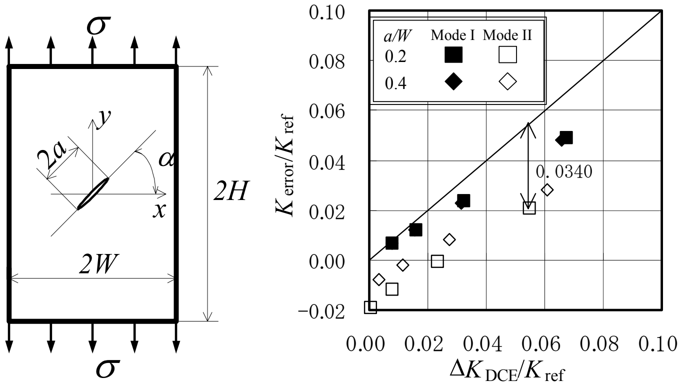

Figure 4: Actual SIF error K_{error} and DCE Index ΔK_{DCE} ($H/W = 2$, m=24, α=30°, $\nu = 0.3$)

The effects of m on mode I and II K_{DCT}s for the case of $a/W = 0.6$ are summarized in Figure 5 to explain why we selected $m = 24$. There are four data for each mark in the figure. As we see from the figure, there was a difference in the mode I $\Delta K_{\mathrm{DCE}}/K_{\mathrm{ref}}$ and mode II $\Delta K_{\mathrm{DCE}}/K_{\mathrm{ref}}$ of up to 2.74% for $m = 8$. We thought that the error index $\Delta K_{\mathrm{DCE}}/K_{\mathrm{ref}}$ for each mode should not show such a large discrepancy, because we estimate the SIF and its error only from FEA displacements. Thus, we set a guideline for this mode I and II $\Delta K_{\mathrm{DCE}}/K_{\mathrm{ref}}$ discrepancy to be lower than 1.5% and selected $m = 24$ for the problem under consideration. Note that the mode I $\Delta K_{\mathrm{DCE}}/K_{\mathrm{ref}}$ changed slightly as m increased.

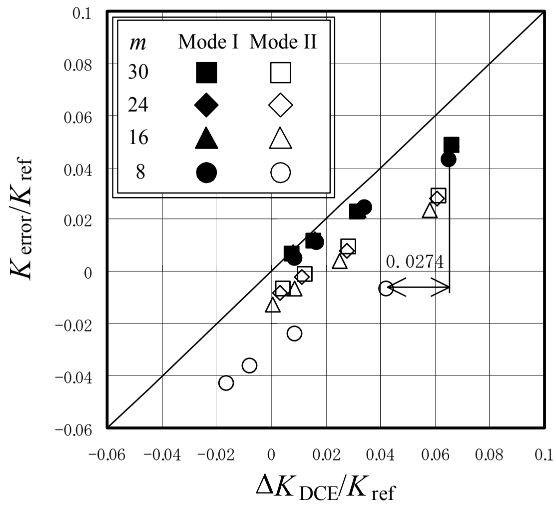

Figure 5: Effect of SE number m on actual SIF error K_{error} and DCE Index ΔK_{DCE} (for $a/W = 0.6$)

As in the right of Figure 2, there are four marks for each a/W in Figure 4 and each mark corresponds to the normalized SE size L/a, which has a positive correlation with ΔK_{DCE}. The maximum difference between K_{error} and ΔK_{DCE} in the figure can be read as 3.40% of K_{ref} for mode II. The figure shows the tendency of ΔK_{DCE} to decrease while L/a is made small and K_{error} to decrease accordingly.

DISCUSSIONS

The DCE Index ΔK_{DCE}, which we proposed in this paper, is intended to give a rough idea of the error of the SIF obtained from FEA results. When we refine the crack tip SEs in both r and θ direction with proper correlation (or in terms of the previous section, decrease L/a and $1/m$), the plot (ΔK_{DCE}, K_{error}) on a plane is expected to approach the origin. That is, ΔK_{DCE} is expected to converge to K_{error}. This characteristic of ΔK_{DCE} is similar to that of the error index proposed in the past [5], [6] which was based on the energy norm. However, we think that ΔK_{DCE} is advantageous because it has the dimension of a SIF and the SIF error can be discussed directly.

When we refine the crack tip SEs, we had better reduce the element size in both r and θ directions properly by correlating two parameters L/a and $1/m$. However, because this makes the finite element division quite difficult, we first fixed the number of elements in the θ direction m and reduced the element size in the r direction in the numerical examples shown in the previous section. The results showed that K_{I} is relatively insensitive to mesh refinements in the θ direction, thus, we can concentrate on refining the mesh in the r

direction once we choose m larger than a certain value. On the other hand, the situation differs with regard to K_{II}; that is, the convergence of K_{II} by varying m should be confirmed. In any case, the validity of m can be judged by the discrepancy between plots on a figure like Figure 5 and the origin when L/a is made small, in case that analytical SIF solutions are known. Note that the m presented in the numerical examples in this paper satisfies this condition (at least for K_I whose accuracy is important for practical problems) and that our error index approximates the SIF error closely for $|\Delta K_{I\,DCE}/K_{I\,ref}| < 0.05$. Next, what about the cases for which the analytical solutions are not known (that is, cases the error index is meant to be developed for)? We think that the results for K_I show (though some more study might be necessary) that when we choose $m \geq 16$ and consider a case under $|\Delta K_{I\,DCE}/K_{I\,DCT}| < 0.05$ instead of $|\Delta K_{I\,DCE}/K_{I\,ref}| < 0.05$, our error index gives an approximate evaluation of the SIF error itself and that there is a possibility to compensate the error in $K_{I\,DCT}$. Regarding to K_{II}, as shown in Figure 5, the plots (K_{error}, ΔK_{DCE}) moves closer to the origin when L/a is made small and m is increased. From this, we expect that an error estimation procedure similar to that for K_I just mentioned can be applied to K_{II}, if we use large m. However, applying large m is not necessarily realistic. In this sense, what we refer to as a SE may not necessarily be suitable for K_{II} evaluation. Nevertheless, if we use this SE for the K_{II} evaluation, a candidate for the m selection criteria is $|\Delta K_{I\,DCE}/K_{I\,ref} - \Delta K_{II\,DCE}/K_{II\,ref}|$ as shown in the previous section, because a discrepancy in accuracy of K_I and K_{II} is not desirable. In situations where the analytical solution is not known (the cases for which the error index was developed), the criteria will be $|\Delta K_{I\,DCE}/K_{I\,DCT} - \Delta K_{II\,DCE}/K_{II\,DCT}|$ instead.

Recently, Rahulkumar et al. [11] proposed an approach to use higher order SEs for an accurate SIF evaluation with a coarse mesh division. However, judging from the results of mixed mode problems discussed in the previous section, it still remains necessary to try to find a proper mesh refinement in the θ direction (selection of m) even though higher order elements are used. For this case too, we think it will be effective to first select m for Barsoum's SE by applying ΔK_{DCE} as proposed in this paper.

CONCLUSIONS

In this paper, an error index for SIF obtained from the FEA results using SEs was developed and was named DCE (Displacement Correlation Error) index. The DCE index was developed as a SIF error index that has the dimension of a SIF, based on the following three facts: (i) The analytical functional form of the crack tip displacements are known. (ii) Though incomplete, displacements on a SE represent a part of the analytical displacement distribution. (iii) The SIF can be evaluated from the displacements of crack tip elements. In spite of the DCE index not being a SIF error itself, the presented numerical results (for the problems whose analytical solutions are known) for appropriate mesh divisions in the θ direction show that the DCE index is close to the actual SIF error, especially for mode I SIF evaluations whose accuracy is important for practical problems and that error compensation might be possible in an engineering sense.

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