

# **SHEAR LAG MODELLING OF THERMAL STRESSES IN UNIDIRECTIONAL COMPOSITES**

Chad M. Landis

Department of Mechanical Engineering and Materials Science, MS 321,  
Rice University, P.O. Box 1892, Houston, TX, 77025  
landis@rice.edu, ph: 713-348-3609, fax: 713-348-5423

## **ABSTRACT**

A two-dimensional shear lag model is presented to analyze the steady state distributions of stress and temperature in unidirectionally reinforced composites. Equations allowing for variations in axial displacement and temperature along any given fiber or matrix region are developed. The derivation of the governing equations is greatly simplified by the assumptions that displacements perpendicular to the fiber direction can be ignored, the axial displacement and temperature are uniform over the cross section of any fiber and the distributions of temperature and axial displacement are bilinear within the matrix regions. These equations are then solved for a configuration with uniformly spaced matrix cracks and used to determine the effective Young's modulus and thermal conductivity of the cracked material.

## **KEYWORDS**

Composites, thermal stresses, shear lag

## **1. INTRODUCTION**

The analysis of stress and strain in multi-fiber composites has been facilitated by the development of shear lag models. These shear lag models treat fibers as one-dimensional load carrying structures that transfer loads to one another through shear stresses within the matrix material. The original multi-fiber shear lag model was formulated by Hedgepeth[1] in 1961. Since then a number of researchers have enhanced this model by including effects like sliding at the fiber matrix interface or the load carrying capability of the matrix, see Landis and McMeeking[2]. A direct two-dimensional extension of the Hedgepeth[1] model was proposed by Beyerlein and Landis[3] to include the effects of matrix stiffness that reduces to the Hedgepeth[1] model in the appropriate limit. The work presented in this paper will build on the model of Beyerlein and Landis[3] to include the effects of steady state temperature distributions. These shear lag

models are most useful in large scale composite failure simulations like those carried out by Ibnabdeljalil and Curtin[4] and Landis *et al.*[5] where more detailed stress calculations would be untenable.

## 2. GOVERNING EQUATIONS

The model system considered is depicted in Figure 1 with the fibers separated by the matrix in a periodic fashion.

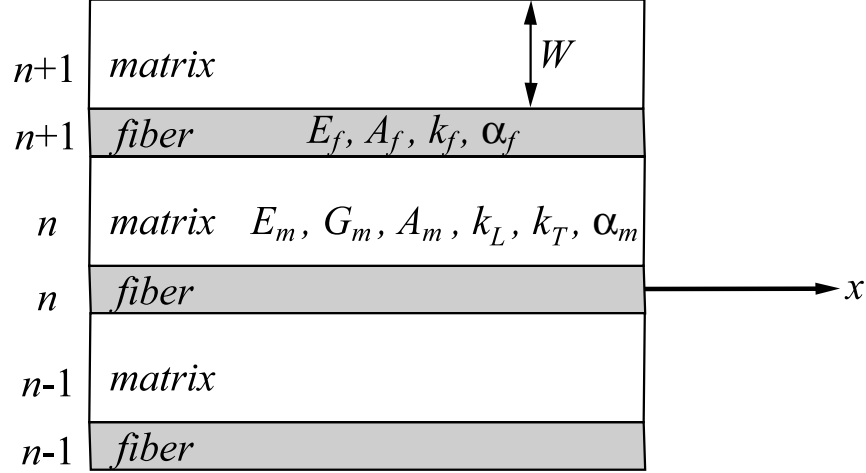


Figure 1: Two dimensional fiber/matrix configuration.

The Young's modulus, area, thermal conductivity and thermal expansion coefficient of the fibers are  $E_f$ ,  $A_f$ ,  $k_f$  and  $\alpha_f$ . The Young's modulus, shear modulus, area, longitudinal thermal conductivity, transverse thermal conductivity, thermal expansion coefficient and width of the matrix regions are  $E_m$ ,  $G_m$ ,  $A_m$ ,  $k_L$ ,  $k_T$ ,  $\alpha_m$  and  $W$ . The thickness of the system is  $t$ , hence  $A_m = Wt$ . Following the procedure outlined by Landis and co-workers [2-3] the equations governing the axial displacement and temperature along the  $n^{\text{th}}$  fiber,  $u_n^f$  and  $T_n^f$ , and the center of the  $n^{\text{th}}$  matrix region,  $u_n^m$  and  $T_n^m$ , are

$$\left(1 + \frac{\eta}{3}\right) \frac{d^2 T_n^f}{dx^2} + \frac{\eta}{12} \frac{d^2 T_n^m}{dx^2} + \frac{\eta}{12} \frac{d^2 T_{n-1}^m}{dx^2} + 2\bar{k}(T_n^m + T_{n-1}^m - 2T_n^f) = 0 \quad (1)$$

$$\frac{\eta}{3} \frac{d^2 T_n^m}{dx^2} + \frac{\eta}{12} \frac{d^2 T_{n+1}^f}{dx^2} + \frac{\eta}{12} \frac{d^2 T_n^f}{dx^2} + 2\bar{k}(T_{n+1}^f + T_n^f - 2T_n^m) = 0 \quad (2)$$

$$\left(1 + \frac{\rho}{3}\right) \frac{d^2 u_n^f}{dx^2} + \frac{\rho}{12} \frac{d^2 u_n^m}{dx^2} + \frac{\rho}{12} \frac{d^2 u_{n-1}^m}{dx^2} + 2\bar{G}(u_n^m + u_{n-1}^m - 2u_n^f) = \left(\alpha_f + \frac{\rho}{3}\alpha_m\right) \frac{dT_n^f}{dx} + \frac{\rho}{12}\alpha_m \left(\frac{dT_n^m}{dx} + \frac{dT_{n-1}^m}{dx}\right) \quad (3)$$

$$\frac{\rho}{3} \frac{d^2 u_n^m}{dx^2} + \frac{\rho}{12} \frac{d^2 u_{n+1}^f}{dx^2} + \frac{\rho}{12} \frac{d^2 u_n^f}{dx^2} + 2\bar{G}(u_n^f + u_{n+1}^f - 2u_n^m) = \frac{\rho}{3}\alpha_m \frac{dT_n^m}{dx} + \frac{\rho}{12}\alpha_m \left(\frac{dT_{n+1}^f}{dx} + \frac{dT_n^m}{dx}\right) \quad (4)$$

$$\rho = \frac{E_f A_f}{E_m A_m}, \quad \eta = \frac{k_f A_f}{k_L A_m}, \quad \bar{G} = \frac{G_m t}{E_f A_f W}, \quad \bar{k} = \frac{k_T t}{k_f A_f W} \quad (5)$$

The essential assumptions required to obtain these equation are that the material is constrained to displace in the  $x$  direction only, shear deformation and transverse temperature gradients in the fibers are neglected and the displacement or temperature profile along the transverse direction in the matrix is bilinear.

### 3. N-INDEPENDENT SOLUTIONS

In general Eqns. (1-4) represent a large set of coupled ordinary differential equations for the displacement and temperature in each fiber and matrix region. However, if the locations of imperfections, i.e. fiber or matrix cracks, are distributed uniformly over all fiber or matrix regions then Eqns. (1-4) reduce to four governing ordinary differential equations. In other words if the distributions of displacement and temperature are independent of the fiber or matrix region number  $n$  then the following equations govern the system:

$$\left(1 + \frac{\eta}{3}\right)T_f'' + \frac{\eta}{6}T_m'' + 4\bar{k}(T_m - T_f) = 0 \quad (6)$$

$$\frac{\eta}{3}T_m'' + \frac{\eta}{6}T_f'' + 4\bar{k}(T_f - T_m) = 0 \quad (7)$$

$$\left(1 + \frac{\rho}{3}\right)u_f'' + \frac{\rho}{6}u_m'' + 4\bar{G}(u_m - u_f) = \left(\alpha_f + \frac{\rho}{3}\alpha_m\right)T_f' + \frac{\rho}{6}\alpha_m T_m' \quad (8)$$

$$\frac{\rho}{3}u_m'' + \frac{\rho}{6}u_f'' + 4\bar{G}(u_f - u_m) = \frac{\rho}{3}\alpha_m T_m' + \frac{\rho}{6}\alpha_m T_f' \quad (9)$$

where  $T_f$ ,  $T_m$ ,  $u_f$  and  $u_m$  are the temperature and displacement distributions in all fibers and all matrix regions. The ' and the '' denote first and second derivatives with respect to  $x$ . The general solutions to Eqns. (6-9) are

$$T_m = T_0 + q_0x + C_{T1}e^{\beta_T x} + C_{T2}e^{-\beta_T x} \quad (10)$$

$$T_f = T_0 + q_0x - \frac{\eta}{\eta+2}C_{T1}e^{\beta_T x} - \frac{\eta}{\eta+2}C_{T2}e^{-\beta_T x} \quad (11)$$

$$u_m = u_0 + \frac{\alpha_f + \rho\alpha_m}{\rho+1}T_0x + \varepsilon_0x + \frac{\alpha_f + \rho\alpha_m}{2(\rho+1)}q_0x^2 + C_{m1}e^{\beta_T x} - C_{m2}e^{-\beta_T x} + C_{u1}e^{\beta_u x} + C_{u2}e^{-\beta_u x} \quad (12)$$

$$u_f = u_0 - \frac{(\alpha_f - \alpha_m)\rho}{8\bar{G}(\rho+1)}q_0 + \frac{\alpha_f + \rho\alpha_m}{\rho+1}T_0x + \varepsilon_0x + \frac{\alpha_f + \rho\alpha_m}{2(\rho+1)}q_0x^2 + C_{f1}e^{\beta_T x} - C_{f2}e^{-\beta_T x} - \frac{\rho}{\rho+2}C_{u1}e^{\beta_u x} - \frac{\rho}{\rho+2}C_{u2}e^{-\beta_u x} \quad (13)$$

$$C_{f1,2} = C_{T1,2} \frac{\eta\alpha_f - \rho\alpha_m - \frac{\beta_T^2\eta\rho}{48\bar{G}}(4\alpha_f + \rho\alpha_m)}{\beta_T(\eta+2) \left[ \frac{\beta_T^2\rho}{48\bar{G}}(\rho+4) - (\rho+1) \right]} \quad (14)$$

$$C_{m1,2} = C_{T1,2} \frac{\eta\alpha_f - \rho\alpha_m + \frac{\beta_T^2 \eta \rho}{48G} [2\eta\alpha_f + \alpha_m \eta(\rho+2) + 2\alpha_m(\rho+4)]}{\beta_T(\eta+2) \left[ \frac{\beta_T^2 \rho}{48G}(\rho+4) - (\rho+1) \right]} \quad (15)$$

$$\beta_T = \left( \frac{48k_T(\eta+1)}{k_L W^2(\eta+4)} \right)^{1/2}, \quad \beta_u = \left( \frac{48G_m(\rho+1)}{E_m W^2(\rho+4)} \right)^{1/2} \quad (16)$$

where  $T_0$ ,  $q_0$ ,  $u_0$ ,  $\varepsilon_0$ ,  $C_{T1}$ ,  $C_{T2}$ ,  $C_{u1}$  and  $C_{u2}$  must be determined from boundary conditions.

The general solution listed in Eqns. (10-16) can be used to determine the effective Young's modulus and effective longitudinal thermal conductivity of a composite with a uniform matrix crack spacing. Consider the crack geometry illustrated in Figure 2.

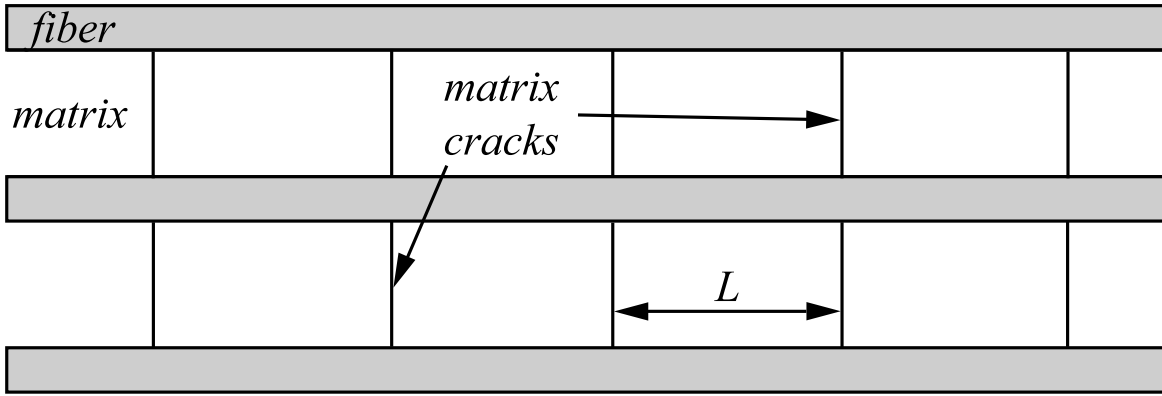


Figure 2: Uniform matrix crack spacing configuration. The matrix cracks are assumed to be traction-free and perfectly insulating.

To analyze the effective thermal conductivity of the cracked composite only the temperature solution needs to be considered and to determine the effective modulus only the displacement solution needs to be considered. Since the distribution of temperature or displacement is bilinear across a matrix region only the *average* heat flux or stress at a matrix crack can be specified to be zero. So for the thermal conductivity calculation the average temperature gradient is set to zero at  $x=0$  and at  $x=L$ . The solution for the temperature distribution in a given segment of material between two matrix cracks is

$$T_f = T_0 + q_0 \left[ x + \frac{\eta}{\beta_T} \frac{\cosh \beta_T x - \cosh \beta_T (x-L)}{\sinh \beta_T L} \right] \quad (17)$$

$$\bar{T}_m = T_0 + q_0 \left[ x + \frac{1}{\beta_T} \frac{\cosh \beta_T (x-L) - \cosh \beta_T x}{\sinh \beta_T L} \right] \quad (18)$$

where  $\bar{T}_m = (T_f + T_m)/2$  is the average temperature across the matrix at a given  $x$  location. The effective thermal conductivity of the composite is then just the average heat flux through the composite divided by the average temperature gradient across the same region. Note that care must be taken to include temperature jumps across the matrix cracks in these averages. The effective thermal conductivity of the cracked composite is

$$k_c = \left( \frac{k_f A_f + k_L A_m}{A_f + A_m} \right) / \left( 1 + \frac{2\eta}{\beta_T L} \frac{\cosh \beta_T L - 1}{\sinh \beta_T L} \right) \quad (19)$$

Eqn. (19) is identical to the result obtained by Lu and Hutchinson [6]. From a solution similar to Eqns. (17-18) for the displacements the effective Young's modulus is

$$E_c = \left( \frac{E_f A_f + E_m A_m}{A_f + A_m} \right) / \left( 1 + \frac{2\rho}{\beta_u L} \frac{\cosh \beta_u L - 1}{\sinh \beta_u L} \right) \quad (20)$$

Notice that when the crack spacing,  $L$ , goes to infinity then the composite conductivity or modulus is the area fraction weighted average of the fibers and matrix. When  $L$  goes to zero then the conductivity and modulus of the composite are  $k_c = k_f A_f / (A_f + A_m)$  and  $E_c = E_f A_f / (A_f + A_m)$ .

Another simple solution for the uniform matrix crack configuration is the distribution of stress in the fibers and matrix when the temperature of the composite,  $T$ , is uniform. The fiber stress,  $\sigma_f$ , and the average matrix stress,  $\bar{\sigma}_m$ , are

$$\sigma_f = E_f (\alpha_m - \alpha_f) T \frac{\rho}{\rho + 1} \left[ 1 + \frac{\sinh \beta_u (x - L) - \sinh \beta_u x}{\sinh \beta_u L} \right] \quad (21)$$

$$\bar{\sigma}_m = E_m (\alpha_f - \alpha_m) T \frac{1}{\rho + 1} \left[ 1 + \frac{\sinh \beta_u (x - L) - \sinh \beta_u x}{\sinh \beta_u L} \right] \quad (22)$$

Notice since there is no applied load, only temperature, that  $\sigma_f A_f + \bar{\sigma}_m A_m = 0$ .

#### 4. DISCUSSION

The simple solutions presented in Eqns. (10-16) do not exploit the true value of the shear lag model. Generally, solutions to the equations of thermoelasticity could be obtained with series solutions for the simple geometry considered in Figure 2 leading to more accurate estimates of the composite conductivity and modulus. However, it is difficult to obtain exact solutions for the distributions of stress and temperature around even a single fiber break. Contrarily, simple superposition techniques exist that can solve Eqns. (1-5) for any finite number of arbitrarily located fiber or matrix cracks. Hence, the simplified shear lag model is most useful in large failure simulations like those carried out in [4] and [5] where rapid solutions for the interactions of cracks with one another and the loading are important.

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