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SEMI-WEIGHT FUNCTION METHOD OF THREE-DIMENSIONAL PROBLEM IN FRACTURE MECHANICS

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ABSTRACT

In two-dimensional fracture analysis, Bueckner proposed a weight function method to solve several problems efficiently. However, there are many difficulties and complexities because the analytical expression of the weight function can not be obtained easily. Liu put forward a concept of semi-weight function and got analytical expression of stress intensity factors and semi-weight functions and satisfactory results in plane problems. Because of the complexity of three-dimensional fracture problems, finite element method and other numerical methods are commonly used, the weight function method can not be used directly and often be used in analysis and calculation after some engineering simplification. The semi-weight function method is used and developed in this paper to solve three dimensional fracture problems, as an extension from plane problems. This paper has contributions as below:

1. From principle of virtual work(reciprocal work theorem), analytical expression of the relationship between semi-weight functions and fracture parameters is obtained through strict theoretical derivation;
2. Analytical expression of semi-weight functions is obtained. A few conditions are satisfied on average;
3. SIF for mode I in three-dimensional problems of plates is analyzed and calculated with use of this method. The stress intensity factors and the distribution across thickness are solved. Relatively exact results are got from calculation example.

The calculation results show that among high precision calculation methods, compared with the weight function method, this method provides applicable analytical expressions of semi-weight functions and in less restrict condition. Compared with finite element method, it needs less amount of calculation.

KEYWORDS

semi-weight function method, stress intensity factors, three-dimensional problems, reciprocal work theorem

INTRODUCTION

The stress intensity factor(SIF) is an important parameter in fracture mechanics. In a three-dimensional finite body, the variation of SIF along the crack front needs to be computed. In most cases, two methods are used to solve typical problems.

One is the finite element method(FEM). With a three-dimensional finite element elastic stress analysis, Raju and Newman[1] calculated the SIF for some commonly used fracture specimens that have a through-the-thickness crack. However, FEM requires great amounts of computation time, especially when large numbers of elements are concentrated near the crack tip. Singular elements, where shape functions have the same stress singularity as the stress field near the crack tip, were developed. The problems are how to construct a singular element and determine the size of singular element.

Another one is the weight function method. Bueckner proposed the weight functions concept in 1970[2]. The weight functions possess many interesting characteristics which make it possible to obtain the SIF by simply calculating an integral along any of the paths around the crack tip. However, in many cases it is more difficult to find the weight functions than the SIF directly.

Liu [3] put forward the semi-weight function method for two dimensional problems in 1991. The semi-functions are independent of boundary conditions and it is only necessary to calculate displacements and tractions along any of the paths around the crack tip. If one uses FEM to calculate displacements and tractions along one of the paths, one need not use many elements near the crack tip and a lot of complex work is saved. This paper considers about how to use this method in calculating Mode-I SIF along the through-the-thickness crack front of finite-thickness fracture specimens.

ANALYSIS

Conclude SIF Expression from Semi-weight Functions

Consider a plate with through-the-thickness crack(figure 1). The thickness of the plate is 2h. In an elastic body, we consider an arbitrary region Ω, including free surfaces and crack face. The boundary of this region is $\partial V = C_s + \Gamma$, where C_S is the crack face and Γ is the other boundary. If we cut out a cylinder with radius

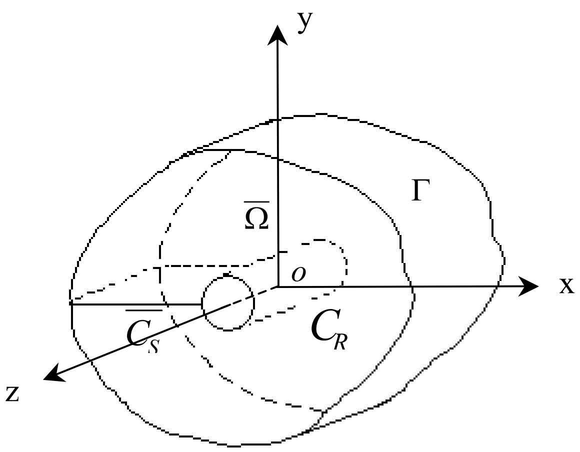

Fig.1: Three-dimensional crack and integration region

R from the crack tip, and the boundary of the cylinder is C_R. The region after cutting out is $\overline{\Omega}$, the crack face is $\overline{C}_S$ and outer boundary is Γ. Then we get

$$\begin{cases} \lim\limits_{R\to 0}\overline{\Omega} = \Omega \\ \lim\limits_{R\to 0}\overline{C}_S = C_S \end{cases}$$

From the principle of virtual work(reciprocal work theorem), we get

$$\int_{\Omega} f_i^{(s)} u_i d\Omega + \int_{\Gamma+\overline{C}_s+C_R} p_i^{(s)} u_i ds = \int_{\Omega} f_i u_i^{(s)} d\Omega + \int_{\Gamma+\overline{C}_s+C_R} p_i u_i^{(s)} ds \tag{1}$$

where (u_i, p_i, f_i) and $(u_i^{(s)}, p_i^{(s)}, f_i^{(s)})$ are two sets of displacements, tractions and volume forces respectively. Expand and transform the above expression along integral path and ignore volume force we get

$$\int_{C_R} (p_i^{(s)} u_i - p_i u_i^{(s)}) ds + \int_{\overline{C}_s} (p_i^{(s)} u_i - p_i u_i^{(s)}) ds = \int_{\Gamma} (p_i u_i^{(s)} - p_i^{(s)} u_i) ds \tag{2}$$

Assume that (u_i, p_i) the real displacements and tractions there must be $p_i = 0$. We set $(u_i^{(s)}, p_i^{(s)})$ as the virtual displacements and tractions which satisfy the conditions of equilibrium equation, stress and strain relationship, $\lim\limits_{r\to 0} u_i = O(r^{-1/2})$ near the crack tip and the traction free on the crack face $p_i^{(s)} = 0$. We name these virtual displacements and tractions semi-weight functions. The above expression can be changed to

$$\int_{C_R} (p_i^{(s)} u_i - p_i u_i^{(s)}) ds = \int_{\Gamma} (p_i u_i^{(s)} - p_i^{(s)} u_i) ds \tag{3}$$

Consider about the singularity near to crack tip, we set

$$u_i^{(s)} = r^{-\frac{1}{2}} f_i^{(s)}(\theta) g_i^{(s)}(z), \tag{4}$$

$$\begin{cases} u = \dfrac{K_I(z)}{8G}\sqrt{\dfrac{2}{\pi}} r^{1/2}((5-8\nu)\cos\dfrac{\theta}{2} - \cos\dfrac{3\theta}{2}) \\ v = \dfrac{K_I(z)}{8G}\sqrt{\dfrac{2}{\pi}} r^{1/2}((7-8\nu)\sin\dfrac{\theta}{2} - \sin\dfrac{3\theta}{2}) \end{cases}, \tag{5}$$

where $f_i^{(s)}$ are functions and not volume forces any more. Take the SIF expression form in plane strain, and change the original K_I to $K_I(z)$. We know, for the through-the-thickness crack problem, physical characteristics in the region at crack tip always behave as in plane strain. Substitute these two expression to left side of equation (3).
We get

$$\lim_{r\to 0} \int_{C_R} (p_i^{(s)} u_i - p_i u_i^{(s)}) ds = \int_{-h}^{h} K_I(z)\Phi(z) dz \tag{6}$$

where $\Phi(z) = \dfrac{1}{4\sqrt{2\pi}(-1+2\nu)}(g_1(z)(\int_{-\pi}^{\pi}(f_1(\theta)h_1(\theta)d\theta + A) + g_2(z)\int_{-\pi}^{\pi}(f_2(\theta)h_2(\theta)d\theta)$

and $\quad h_1(\theta) = \cos\dfrac{\theta}{2}((3-4\nu)^2 + (-5+4\nu)\cos\theta + 2\cos 2\theta)$

$$h_2(\theta) = \sin\frac{\theta}{2}((3-4\nu)^2 + (-1+4\nu)\cos\theta + 2\cos 2\theta)$$

$$A = -8(-1+\nu)(-1+2\nu)(f_1(-\pi) + f_1(\pi))$$

thus expression (1) can be changed to

$$\int_{-h}^{h} K_I(z)\Phi(z)dz = \int_{\Gamma} (p_i u_i^{(s)} - p_i^{(s)} u_i)ds \,. \tag{7}$$

That is the expression of SIF from semi-weight functions.

Three-dimensional Semi-weight Functions

From the process of obtaining SIF, we see that $(u_i^{(s)}, p_i^{(s)})$ satisfy three conditions, that is the conditions of equilibrium equation, stress and strain relationship, $\lim_{r\to 0} u_i = O(r^{-1/2})$ near the crack tip and the traction free on the crack face, $p_i^{(s)} = 0$. Any functions that satisfy these conditions can be regarded as semi-weight functions.

We set displacement functions in semi-weight functions as

$$\begin{cases} u^{(s)} = r^{-\frac{1}{2}} f_1(\theta) g_1(z) \\ v^{(s)} = r^{-\frac{1}{2}} f_2(\theta) g_2(z) \\ w^{(s)} = r^{-\frac{1}{2}} f_3(\theta) g_3(z) \end{cases} , \tag{8}$$

and find that the condition of equilibrium equation can not be strictly satisfied. It means that equilibrium equation, no matter what expressions these functions are, can not be zero. We relax this condition and let the integration of equation along thickness of plate be zero. For the two-dimensional problem, these conditions can be all strictly satisfied[2].

If we select proper functions, other conditions can be satisfied. Finally we get semi-weight functions for three-dimensional problems

$$\begin{cases} u^{(s)} = \dfrac{1}{G} r^{-\frac{1}{2}} \cos\theta \cos\dfrac{m\pi z}{h} \\ v^{(s)} = \dfrac{\nu}{2G(1-\nu)} r^{-\frac{1}{2}} \sin\theta \cos\dfrac{m\pi z}{h} \\ w^{(s)} = 0` \end{cases} , \tag{9}$$

where G is shear modulus and *m* is any natural number.

APPLICATION

As an application, we calculate SIF along the crack front of center-crack tension, which is a classical example in three-dimensional problems, and compare the result of Newman[1]. In our application, we expand $K_I(z)$ in an power series as $K_I(z) = \sum_{i=0}^{m} k_i (\frac{z^2}{h^2})^i$. In this place, the selection of number m is the same as in semi-weight functions. With the selection of different m, equations can be introduced. Solving the equations, we get every k_i and finally $K_I(z)$. When we calculate the integration in the right side of (3), we calculate the actual displacements and stresses with FEM. In our FEM model, no singularity elements were used.

Compare with [1], we select $\frac{b}{a}=0.875$, $\frac{c}{a}=0.5$, $\frac{h}{c}=1.5$, $\nu=\frac{1}{3}$. In our model, we set $a=10m$,

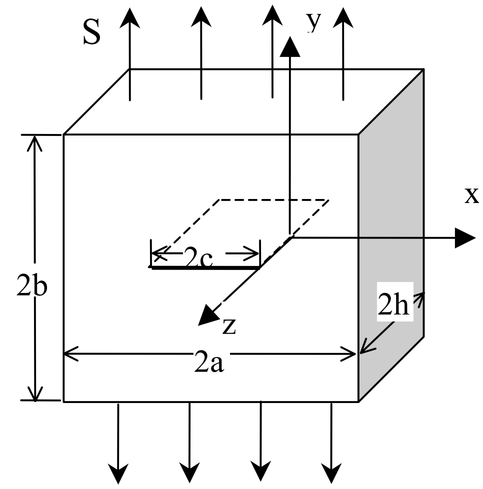

Fig.2: Model of center-crack tension

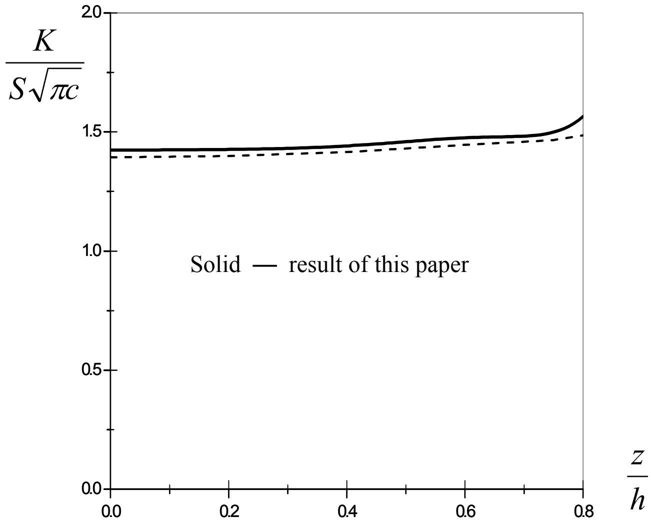

Fig.3: Comparison of SIF between this paper and ref.[1]

$b=8.75m$, $c=5m$, $h=7.5m$, Youngs modulus $E=2.110^{11}pa$, $S=1000pa$, and get the result expression

$$F(\xi)=1.424+0.0709501\xi^2-0.292252\xi^4+5.83877\xi^6-18.2836\xi^8+16.3159\xi^{10}, \tag{10}$$

where $\xi=\frac{z}{h}$, $F=\frac{K_I}{S\sqrt{\pi c}}$

Change the thickness ratio h/c and compare with result of reference[1]

Table 1: Comparison of SIF of various thickness specimens between this paper and refer.[1]

	0.5	1	1.5	2
This paper	1.4781	1.4458	1.4247	1.3910
Newman[1]	1.4833	1.4333	1.401	1.3833
Difference %	-0.35	0.87	1.69	0.56

We can see the result of this paper and ref.[1] are very close. The maximum difference does not exceed 5%

CONCLUSION

This paper extends the semi-weight function method to three-dimensional fracture problems. It deduced three-dimensional SIF expression in a form of integration from semi-weight functions. Analytic expression of three-dimensional semi-weight functions are also introduced. Comparison of application results between this method and classical the FEM results shows that they are very close, the difference is less than 5%. The semi-weight function method is a convenient method for calculating the SIF in various problems. It is only necessary to know the tractions and displacements along any paths around the crack tip. By using this method the complex analysis near the crack tip can be avoided. Even when the approximation of far field is not very accuracy, the semi-weight function method can also get some satisfactory result in engineering application.

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