ICF10079OR

# ON THE CRACK EXTENSION FORCE OF CURVED CRACKS

K. Eriksson

Department of Solid Mechanics, Luleå University of Technology
SE 97187 Luleå (Sweden)

## ABSTRACT

A domain independent integral is obtained from the principle of virtual work. A suitable choice of the virtual displacement field allows variation of the position of a crack tip. For materials possessing a strain energy function Eshelby's [1] definition of the force on a point defect is used to obtain the crack extension force. The method is general and allows treatment of a crack whose surfaces and front are curved by using curvilinear coordinates. To illustrate the applicability of the method three examples of the point-wise crack extension force are given, with different combinations of crack surface and crack front curvature. A general expression of the crack extension force for curved cracks is suggested. In Cartesian coordinates the proposed expression is reduced to Rice's [2] *J*-integral.

## KEYWORDS

Curved crack, Crack extension force, Domain independent integral, Elastic material.

## INTRODUCTION

Eshelby's [1] definition of the force acting on a point defect in an elastic solid is briefly: minus the rate of increase of the total energy with respect to a variation of the position of the defect. The total energy is the sum of the strain energy of the part of the solid considered and of the potential energy of its external loads, if any. The force is expressed as an integral of a normal component of Eshelby's energy momentum tensor taken over a finite and closed surface containing the defect. Two essential features of this integral are that it is path independent and has a finite value as the integration area is shrunk towards zero around the defect. In crack problems the point defect is naturally associated with the tip of the crack and the force integral, which must be zero on stress-free crack surfaces, is called the 'crack extension force'.

In this work we start by constructing a domain independent integral by using the principle of virtual work. A suitable choice of the virtual displacement field allows variation of the position of the crack tip. For materials possessing a strain energy function a general expression of a domain independent integral is derived, which is not associated with any geometry in particular. Application to a given crack geometry is most conveniently done by a suitable choice of coordinate system in which the description of the crack geometry is as simple as possible. The method is general and by choosing curvilinear coordinates, cracks with curved surfaces and front can be treated, the only restriction being that the curvilinear coordinate system must be orthogonal. This follows from the condition that crack extension is confined to the tangent plane of the crack and perpendicular to the crack front tangent and that the domain independent integral must vanish on stress free crack surfaces. The crack extension force is finally obtained by applying Eshelby's [1] definition.

Three examples of curved cracks are given, with different combinations of crack surface and crack front curvature. The first is the penny-shaped crack with radial crack extension (plane crack surface and curved crack front), the second the circular arc crack with crack extension circumferentially (curved crack surface and straight crack front) and the third the latter crack geometry but with crack extension axially (curved crack surface perpendicular to the direction of crack extension). In order to obtain the point-wise value of the crack extension force, loading and crack extension are assumed uniform along the crack front in all cases. The results imply the general form of the crack extension force for a crack with curved surfaces and a curved front. For a straight crack in Cartesian coordinates in two dimensions the general form of the crack extension force is reduced to Rice's [2] *J*-integral.

## A DOMAIN INDEPENDENT INTEGRAL

Let $\theta_i$, $i = 1,2,3$ be orthogonal curvilinear coordinates and consider the integral

$$\delta I = \oint_S t_i \delta u^i \, dS - \int_V \delta W \, dV \tag{1}$$

where $\delta$ denotes variation and *S* and *V* are surface and volume of a body or any part thereof, respectively, $t_i$ stress covector, $u^i$ displacement and *W* strain energy. Superscripts denote contravariant and subscripts covariant tensor properties. Summation over indices appearing twice is implied.

Assume that there exists a strain energy function $W = W(u^i_{,j})$ such that $\tau_i^j = \partial W / \partial u^i_{,j}$, where $\tau^i_j$ is stress and a comma (,) denotes covariant differentiation. Take the virtual displacement $\delta u^i$ as minus the total differential of the "actual" $u^i$, (the solution to the problem studied), that is

$$\delta u^i = -u^i_{,k} \delta\theta^k , \tag{2}$$

where $\delta\theta^k$ can be arbitrarily chosen. By the principle of virtual work the integral $\delta I$ is zero for any regular region. Further, (2) allows modelling of crack extension in the direction of $\theta_i$ through variation of the position of the crack tip.

Using the facts that i) variation and differentiation and ii) the second covariant derivative are commutative, the internal virtual work $\delta W$ can now be written

$$\delta W = -\frac{\partial W}{\partial \theta^k} \delta\theta^k - \tau_i^j u^i_{,k} \delta\theta^k_{,j} \tag{3}$$

Inserting in (1) yields

$$\delta I = \int_V \frac{\partial W}{\partial \theta^k} \delta\theta^k \, dV - \oint_S t_i u^i_{,k} \delta\theta^k \, dS + \int_V \tau_i^j u^i_{,k} \delta\theta^k_{,j} \, dV \tag{4}$$

As *V* is arbitrary, the integral expression $\delta I$ is domain independent if *V* is regular. Further, by using Cauchy's formula $t_i = \tau_i^j n_j$, where $n_j$ is the outward positive unit normal vector on *S*, the divergence theorem, the equations of equilibrium, $\tau^j_{i,j} = 0$, then, as expected, (3) can be obtained from the integrand of the surface integral in (1). Eqn. (4) is quite general and not associated with any geometry in particular. Application to a given crack geometry is most conveniently done through a suitable choice of coordinate system, as shown in the examples.

**The penny-shaped crack**

Let $r, \theta, z$ be cylinder coordinates. The non-zero Christoffel symbols are $\Gamma^{r}_{\theta\theta} = -r$ and $\Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} = 1/r$. Let $a$ and $\Psi$ be suitable constants, $r \le a$, $z = 0$ the crack plane and $\delta r$ an increment which is independent of $\theta_i$. At $r = a$ in particular, the crack extension increment is $\delta r$. Also, let the loading be axisymmetric. In the last integral in (4) the only non-zero component of $\delta\theta^{k}_{,j}$ is $\delta\theta^{\theta}_{,\theta} = \delta r/r$. Writing the variation $\dfrac{\partial W}{\partial \theta^{k}}\delta\theta^{k} = \dfrac{\partial W}{\partial r}\delta r$ we obtain from (4)

$$\delta I = \int_V \frac{\partial W}{\partial r}\delta r\, dV - \oint_S t_i u^{i}_{,r}\delta r\, dS + \int_V t^{\theta}_{i} u^{i}_{,\theta}\frac{\delta r}{r} dV \tag{5}$$

Adding and subtracting $W/r$ to and from the first and second volume integral and writing the first volume integral as a surface integral, we get

$$\delta I = \oint_S \left(W n_r - t_i u^{i}_{,r}\right)\delta r\, dS - \int_V \frac{1}{r}\left(W - t^{\theta}_{i} u^{i}_{,\theta}\right)\delta r\, dV \tag{6}$$

Let $\Gamma_e$ and $\Gamma_o$ be curves in the $r-z$-plane which enclose the crack tip, $\Gamma_e$ infinitely close to and $\Gamma_o$ remote from the crack tip and $V$ a tubular volume generated by rotation of $\Gamma_e$ and $\Gamma_o$ in the sector $|\theta| \le \Psi/2$ around the z-axis. The surface $S$ which surrounds a finite sector of the plane axisymmetric crack is divided into six parts; those generated by $\Gamma_e$ and $\Gamma_o$, here denoted $S_e$ and $S_o$, respectively, the crack surfaces and both ends of the tubular volume.

It is easily seen that the surface integral in (6) vanishes on the crack surfaces and the ends of the tubular volume. On the crack surfaces $n_r = 0$ and because they are stress free, $t_i = 0$. On the ends of the tubular region $n_r = 0$ and on account of axisymmetric loading and deformation $t_r = t_z = 0$ and $u_{\theta,r} = 0$ so that $t_i u^{i}_{,r} = 0$ there. Further, the only non-zero $t^{\theta}_{i}$-component in the volume integral is $t^{\theta}_{\theta}$.

The surface element on $S_e$ and $S_o$ can be written $dS = r\,\Psi\, d\Gamma$, where $d\Gamma$ is a curve element.
Take $\Gamma_e$ and $\Gamma_o$ positive counterclockwise on $S_e$ and $S_o$, respectively. Then Eqn. (2), taking $\delta r$ outside the integrals and writing the volume element $dV = \Psi\, r\, dA$, where $dA$ is an area element and noting that all integrands are independent of $\theta$, yield, as $\Gamma_e$ is shrunk towards zero around the crack tip,

$$\begin{aligned}\delta I = &-\Psi a \delta r \lim_{\Gamma_e \to 0} \oint_{\Gamma_e}\left(W n_r - t_i u^{i}_{,r}\right) d\Gamma + \Psi \delta r \oint_{\Gamma_o}\left(W n_r - t_i u^{i}_{,r}\right) r\, d\Gamma \\ &- \Psi \delta r \int_{A_o}\left(W - t^{\theta}_{\theta} u^{\theta}_{,\theta}\right) dA\end{aligned} \tag{7}$$

where $A_o$ is the area inside $\Gamma_o$.
Now $\delta I = 0$ implies a variation of the total energy of the domain considered in accordance with the definition of Eshelby [1]. As $\delta r$ is the virtual displacement of the crack tip

$$F = \lim_{\Gamma_e \to 0} \oint_{\Gamma_e}\left(W n_r - t_i u^{i}_{,r}\right) d\Gamma \tag{8}$$

must be the associated crack tip or crack extension force per unit length of the crack front. Cancelling common factors Eqs. (7) and (8) yield

$$F = \oint_{\Gamma_o}\left(W n_r - t_i u^{i}_{,r}\right) d\Gamma - \frac{1}{a}\int_{A_o}\left(W - t^{\theta}_{\theta} u^{\theta}_{,\theta}\right) dA \tag{9}$$

Taking physical components ($\sigma_{ij}$, $\varepsilon_{ij}$, $u_i$, etc.) and using $\varepsilon_{\theta\theta} = u_{\theta,\theta}$ we get

$$F = \oint_{\Gamma_o} \left( W n_r - \tau_i u_{i,r} \right) d\Gamma - \frac{1}{a} \int_{A_o} \left( W - \sigma_{\theta\theta} \varepsilon_{\theta\theta} \right) dA \tag{10}$$

per unit crack front length circumferentially, which is also the point-wise value of the crack extension force.

## The circular arc crack

Let $r, \theta, z$ be cylinder coordinates as above, $b$ an additional constant and $r = a$, $|\theta| \le \Psi / 2$ and $|z| \le b/2$ the crack plane. We consider crack extension in two different directions; in the first, $\delta\theta = a\delta\theta_\alpha / r$, where $\delta\theta_\alpha$ is independent of $\theta_i$, is an increment in general. At the crack tip $\theta = \Psi / 2$ the crack extension is $a\delta\theta_\alpha$. The only non-zero component of $\delta\theta^k_{,j}$, which in this case is $\delta\theta^r_{,\theta} = -r\delta\theta$ and the variation $\frac{\partial W}{\partial \theta^k}\delta\theta^k = \frac{1}{r}\frac{\partial W}{\partial \theta} r\delta\theta$ yield with (4)

$$\delta I = \int_V \frac{1}{r}\frac{\partial W}{\partial \theta} r\,\delta\theta\, dV - \oint_S t_i u^i_{,\theta}\, \delta\theta\, dS - \int_V \tau^\theta_i u^i_{,r}\, r\delta\theta\, dV \tag{11}$$

Transforming the first volume integral into a surface integral, we get

$$\delta I = \oint_S \left( W n_\theta - t_i u^i_{,\theta} \right) \delta\theta\, dS - \int_V \tau^\theta_i u^i_{,r}\, r\delta\theta\, dV \tag{12}$$

Let $V$ be a tubular volume along the crack tip considered and with plane ends in the $z$-direction. For simplicity we consider uniform loading along the crack front. In this case also, the surface integral in (12) is zero on both ends of the tube and in particular on the curved crack surfaces. The surface element can be written $dS = b\,d\Gamma$ and the volume element $dV = b\,dA$. With suitable redefinition of $\Gamma_e$ etc., we arrive at the crack extension force, noting that $a\delta\theta_\alpha$ is its physical displacement, with arguments similar to those for the penny-shaped crack

$$F = \oint_{\Gamma_o} \frac{1}{r}\left( W n_\theta - t_i u^i_{,\theta} \right) d\Gamma - \int_{A_o} \left( \tau^\theta_r u^r_{,r} + \tau^\theta_\theta u^\theta_{,r} \right) dA \tag{13}$$

In physical components (13) reads

$$F = \oint_{\Gamma_o} \left( W n_\theta - \tau_i u_{i,\theta} \right) d\Gamma - \int_{A_o} \frac{1}{r}\left( \sigma_{r\theta} u_{r,r} + \sigma_{\theta\theta} u_{\theta,r} \right) dA \tag{14}$$

per unit length of the crack front (in the z-direction).

Finally, we consider crack extension in the $z$-direction of the circular arc crack. Let $\delta z$ be an increment that is independent of $\theta_i$. The crack extension increment at $z = b/2$ is thus $\delta z$. Now, all $\delta\theta^k_{,j} = 0$ and the variation $\frac{\partial W}{\partial \theta^k}\delta\theta^k = \frac{\partial W}{\partial z}\delta z$ yield with (4)

$$\delta I = \int_V \frac{\partial W}{\partial z}\delta z\, dV - \oint_S t_i u^i_{,z}\, \delta z\, dS \tag{15}$$

Let $V$ be a tubular volume around the crack tip considered and otherwise identical to that of the penny-shaped crack. The surface and volume element can in this case be written $dS = \Psi r\, d\Gamma$ and $dV = dr\, r\Psi\, dz$, respectively. Transforming the volume integral we get

$$\delta I = \Psi \oint_S \left( W n_z - t_i u^i_{,z} \right) r\delta z\, d\Gamma \tag{16}$$

The surface integral (16) is with arguments similar to the above cases zero on the curved crack surface. For uniform loading along the crack front the surface integral also vanishes on the ends of the tubular volume.

As $\Gamma_e \to 0$ implies that $r \to a$ it follows from Eqn. (8) that

$$\lim_{\Gamma_e \to 0} \oint_{\Gamma_e} \left( W n_z - t_i u^i_{,z} \right) r \, d\Gamma = aF \tag{17}$$

Taking $\delta I = 0$ and cancelling common factors Eqs. (16) and (17) yield

$$aF = \oint_{\Gamma_o} \left( W n_z - t_i u^i_{,z} \right) r \, d\Gamma \tag{18}$$

The integrand in (18) contains a singularity at the crack tip. This means that $r = a$ can be extracted from the integral (formal proof omitted here, compare however with the contour integral of the penny-shaped crack, Eqn. (10)) and cancelled. In physical components the crack extension force is

$$F = \oint_{\Gamma_o} \left( W \mathrm{n}_z - \mathrm{t}_i \mathrm{u}_{i,z} \right) d\Gamma \tag{19}$$

per unit length of the crack front circumferentially.

## DISCUSSION

In the examples given we have considered uniform crack extension and loading only, in order to enhance the influence of curvature of the surface of the crack and of the crack front and determined a point-wise value of the crack extension force. In a general loading case the surface integrals on the plane ends of the tubular region around the crack tip (taken together) do not vanish. Because the surface integral contains a covariant derivative its value is dependent on curvature. However, the surface integrals in question are also different from zero for a plane crack with a straight crack front in a general loading case, expressed in Cartesian coordinates, as shown by Carpenter et al. [3]. This feature is thus not exclusive of curved cracks.

The increment $\delta q^i$ in (2) can be arbitrarily chosen. In curvilinear coordinates the covariant derivative $\delta q^i_{,j}$ is however not necessarily always zero, even if $\delta q^i$ is independent of $q_j$, although such cases can in fact be found. The last integral in (4) is thus in general not zero for curved cracks and this term is the source of the "correction" terms to the well-known remaining part of the expression for the crack extension force, the contour integral. In Cartesian coordinates the covariant derivative reduces to its partial counterpart and all $\partial \delta q^i / \partial q^j$ are zero. The integral (4) is reduced to

$$\delta I = \int_V \delta W \, dV - \oint_S t_i u_{i,k} \delta q_k \, dS \tag{20}$$

and for crack extension e.g. in the $x$-direction in the $x$-$y$-plane results

$$F = \oint_{\Gamma_o} \left( W \mathrm{n}_x - \mathrm{t}_i \mathrm{u}_{i,x} \right) d\Gamma \tag{21}$$

The crack surface of the penny-shaped crack is plane. The surface integral in (10) is thus a correction term due to the curvature of the crack front. In this case the curvature is constant along the crack front. It is also seen in (7) that the correction term is constant per unit length circumferentially. The result (10) has been obtained previously by many, as recently reviewed by Eriksson [4].

For the circular arc crack and crack extension circumferentially the crack front is straight but the crack plane is curved. The area integral in (14) is in this case a correction term for the crack plane curvature in the direction of crack extension. An exact analytical solution of the stress intensity factors of the circular arc crack has been obtained by Cotterell and Rice [5]. Lorentzon and Eriksson [6] have found the effective stress intensity factor calculated from Eqn. (14) using results of a finite element analysis to be within 1 percent on the average from those of the analytical solution, for identical boundary conditions.

In the last example, the circular arc crack and crack extension axially, both the crack surface and the crack front are straight in the direction of crack propagation. In this case there is no correction term. We note however, that the crack plane is curved in a direction perpendicular to the direction of crack extension.

From the above results we conclude that the general expression for the point-wise crack extension force of a curved crack must be a sum of a contour integral and one or more area integrals. The contour integral corresponds to or can be reduced to (21). There are two types of area integrals, one that is due to the curvature of the crack front in the direction of crack extension and the other that is due to the curvature of the crack surface in the direction of crack extension. The contribution to the total crack extension force from both types of volume integral increases with curvature. The crack surface curvature in a direction perpendicular to the direction of crack extension does not affect the crack extension force.

If the crack front radius *a* in (10) tends to infinity the contribution from the correction term tends to zero and any finite segment of the crack front approaches a straight line. Similarly, if the radius *r* in (14) tends to infinity the contribution from the correction term tends to zero and the crack surface approaches a plane. All expressions for the crack extension force thus tend to (21) as curvature diminishes. We recognise of course this expression as identical to Rice's [2] *J*-integral in two dimensions.

## References


1. Eshelby, J.D. (1975). *J. Elast*. 5, 321-335.
2. Rice, J.R. (1986). *J. Appl. Mech.* 35, 379-386.
3. Carpenter, W.C., Read, D.T. and Dodds, R.H. (1986). *Int. J. Frac.* 31, 303-323.
4. Eriksson, K. (1998). *Int. J. Fract.* 91, L31-L36.
5. Cotterell, B. and Rice, J.R. (1980). *Int. J. Frac.* 16, 155-169
6. Lorentzon, M. and Eriksson, K. (2000). *Engng. Frac. Mech.* 66, 423-439.

16 May 2001

Tracy Collier, ICF10 Conference Organiser
ICF10 Secretariat
Elsevier Science
The Boulevard
Langford Lane
Kidlington
Oxford OX5 1GB
U.K.

Re: Manuscript Number: ICF10079OR

According to Prof Karihaloo's instructions in his letter 9th May 2001, I enclose a) two hard copies of my manuscript ”On the crack extension force of curved cracks”, b) an electronic version on disk (Windows 95, World 7.0), c) a copyright form and d) a hard copy of the extended abstract.

The manuscript and the extended abstract are revised in that I have corrected some misprints, etc. There are no figures. The hard copies of the manuscript and the electronic version are all identical.

Kind regards

Kjell Eriksson