MATHEMATICAL MODEL OF A CRACK ON AN IMPERFECT INTERFACE

Y.A.Antipov[1] and A.B.Movchan[2]

[1]Department of Mathematical Sciences, The University of Bath,
Bath BA2 7AY, UK
[2]Department of Mathematical Sciences, The University of Liverpool,
Liverpool L69 3BX, UK

ABSTRACT

A mathematical model of a crack along a thin soft interface layer is studied in this paper. This type of interface could arise in a ceramic support that has been coated with a layer of high surface area material which contains the dispersed catalyst. Asymptotic analysis is applied to replace the interface layer with a set of effective contact conditions. We use the words "imperfect interface" to emphasise that the solution (the displacement field) is allowed to have a non-zero jump across the interface. Compared to classical formulations for cracks in dissimilar media (where ideal contact conditions are specified outside the crack), in our case the gradient field for the displacement is characterised by a weak logarithmic singularity. The scalar case for the Laplacian operator as well as the vector elasticity problem are considered. Numerical results are presented for a two-phase elastic strip containing a finite crack on an imperfect interface.

KEYWORDS

Imperfect interface, crack, Wiener-Hopf problem, integral equation

INTRODUCTION

Motivation for this work arises from the study of the fracture of ceramic catalytic monolith combustors that are being incorporated into new proto-type designs of gas turbines. The ceramic monolith consists of an extruded structure that contains a large number of parallel channels, e.g. consisting of: 62 cells/cm^2, each cell 1.1mm×1.1mm square, with an open frontal area of 66%. The ceramic surface is coated with a high surface area material (e.g. γ-Al_2O_3) which contains the dispersed catalyst. It is in the catalytic layer (also known as the wash-coat), where the combustion reactions take place (e.g. $CH_4+2O_2 \rightarrow CO_2+2H_2O$), and the energy associated with this highly exothermic reaction is released. In the application in a gas turbine combustor, temperatures of the catalyst layer could vary from ambient conditions (when the turbine is not working) up to 1100°C. It is important that the catalyst layer remains firmly bound to the ceramic support structure during this process. If it cracks and shears, then catalyst will be lost from the monolith which would a) result in a lost in performance of the combustor, and b) lead to possible damage of components downstream of the combustor. Further information on catalytic combustion and layer of catalyst, this gives a two-phase structure. In examining the surface, cracks are clearly visible in the layer. The cracks would have occurred as a result of a) shrinkage of the coated layer (after drying and calcining), and b) differences in coefficients of thermal expansion as the material was exposed to a wide range of temperatures. The presence of a crack on the surface is not considered necessarily to be a problem, however, if the crack propagates and the interface is sheared then this will lead to catalyst loss.

It is documented in engineering literature that the damage of ceramic structures is accompanied by "crack bridging". In the model presented here we assume that the bridging effect exists along the whole interface surface between the substrate and the layer of catalyst (we shall also use the words "imperfect interface" or "soft adhesive"), and, in addition, a crack with zero tractions on its faces is introduced along the interface contour. We study the problems of anti-plane shear and elasticity problems for this two-phase structure. Mathematical models of interfacial cracks are well-developed in the literature for the cases when ideal contact conditions prevail on an interface surface outside a crack. Asymptotic models of elastic adhesive joints were introduced in [2]. The adhesive was modelled as a thin soft layer where effective contact conditions involve continuity of tractions and a linear relation between the traction components and the displacement jump across the adhesive. Laminated structures with linear interfaces were also studied in [3].

In the present work we analyse mathematical models of cracks along imperfect interface boundaries and make an emphasis on the asymptotic behaviour of the solution and its derivatives near the crack ends and at infinity. In contrast to the results already published in the literature, on the interface boundary (outside the crack) we allow for a non-zero displacement jump specified as a function of traction components. The presence of this condition affects the asymptotics of the displacement and stress components in the vicinity of the crack ends.

ANTI-PLANE PROBLEM FOR A STRIP WITH A SEMI-INFINITE CRACK ALONG AN IMPERFECT INTERFACE

Consider two bodies Ω_+ and Ω_- connected through a thin interface layer Ω_0 of thickness ϵ. Assume that the material occupying Ω_+,Ω_- and Ω_0 is characterised by the shear moduli μ_+, μ_- and $\mu_0 = \epsilon\mu$ respectively, where μ is of the same order as μ_+ and μ_-. For the case of anti-plane shear, the displacements u^+, u^- and $u^{(0)}$ in Ω_+,Ω_- and Ω_0 satisfy the Laplace equation in the corresponding strips and the following boundary conditions

$$u^{\pm} = u^{(0)}, \quad \mu_{\pm}\frac{\partial u^{\pm}}{\partial y} = \mu_0 \frac{\partial u^{(0)}}{\partial y} \quad \text{on} \quad \Gamma_{\pm} = \{(x, y) : y = \pm\epsilon/2\} .$$

Let $t = \epsilon^{-1}y$, so that within Ω_0, $|y| < \epsilon/2$, and hence $|t| < 1/2$. In terms of x and t

$$\nabla^2 u^{(0)} = \epsilon^{-2}\frac{\partial^2 u^{(0)}}{\partial t^2} + \frac{\partial^2 u^{(0)}}{\partial x^2} = 0.$$

Denote by $u_0^{(0)}$ the leading term of $u^{(0)}$. Then to leading order, the tractions are continuous across the interface layer and proportional to the displacement jump $\chi(x) = u(x, +0) - u(x, -0)$:

$$\mu_+\frac{\partial u}{\partial y}|_{y=+0} = \mu_-\frac{\partial u}{\partial y}|_{y=-0} = \mu\chi(x), \quad |x| < \infty.$$

First, we analyse the Dirichlet problem. Assume that a three-phase strip contains a semi-infinite delamination crack $\{-\infty < x < 0, y = \pm 0\}$. Ahead of the crack, there exists a thin layer of soft adhesive.

Formally, the problem is set as follows

$$\nabla^2 u(x,y) = 0, \quad |x| < \infty, \quad -b < y < 0, \quad 0 < y < a,$$

$$u(x,a) = u(x,-b) = 0, \quad |x| < \infty,$$

$$\mu_\pm \frac{\partial u}{\partial y}|_{y=\pm 0} = \mu\chi(x), \quad x > 0; \quad \mu_\pm \frac{\partial u}{\partial y}|_{y=\pm 0} = p(x), \quad x < 0. \tag{1}$$

where $p(x)$ characterises the shear load applied to the crack faces. To reduce the boundary-value problem (1) to a Wiener-Hopf functional equation, we extend the second boundary condition in (1) to the whole real axis:

$$\mu_\pm \frac{\partial u}{\partial y}(x,\pm 0) = \mu\chi(x) + \psi(x), \quad |x| < \infty, \tag{2}$$

where $\psi(x)$ is unknown as $x < 0$ and $\psi(x) = 0$ as $x > 0$. Applying the Fourier transform to the problem (1), using the condition (2) and introducing the integrals

$$\Phi^-(\alpha) = \int_{-\infty}^0 \psi(\xi) e^{i\alpha\xi} d\xi, \quad \Phi^+(\alpha) = \int_0^\infty \chi(\xi) e^{i\alpha\xi} d\xi, \quad P^-(\alpha) = \int_{-\infty}^0 p(\xi) e^{i\alpha\xi} d\xi,$$

we get the following Wiener-Hopf equation

$$\Phi^-(\alpha) = G(\alpha)[\mu\Phi^+(\alpha) + P^-(\alpha)], \quad -\infty < \alpha < +\infty, \tag{3}$$

$$G(\alpha) = 1 + \frac{\mu}{\alpha}\left(\frac{\tanh \alpha a}{\mu_+} + \frac{\tanh \alpha b}{\mu_-}\right).$$

Since the function $G(\alpha)$ is even and has zero increment of $\arg G(\alpha)$ along the real axis, the solution of the problem (3) is defined by the following formulae

$$\Phi^-(\alpha) = -\frac{\Psi^-(\alpha)}{X^-(\alpha)}, \quad \alpha \in \mathbb{C}^- \quad , \quad \Phi^+(\alpha) = -\frac{\Psi^+(\alpha)}{\mu X^+(\alpha)}, \quad \alpha \in \mathbb{C}^+,$$

where $\mathbb{C}^+ = \{\alpha : \mathrm{Im}(\alpha) > 0\}$, $\mathbb{C}^- = \{\alpha : \mathrm{Im}(\alpha) < 0\}$, $\Psi^\pm(\alpha)$ and $X^\pm(\alpha)$ are the limit values of the functions

$$\Psi(\alpha) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{X^+(\beta)P^-(\beta)}{\beta - \alpha} d\beta,$$

$$X(\alpha) = \exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{+\infty} \ln G(\beta) \frac{d\beta}{\beta - \alpha}\right\} = \exp\left\{\frac{\alpha}{\pi i}\int_0^{+\infty} \ln G(\beta) \frac{d\beta}{\beta^2 - \alpha^2}\right\}, \quad \alpha \in \mathbb{C} \backslash \mathbb{R}^1.$$

Note that the function $G(\alpha)$ is bounded at the origin and tends to 1 as $\alpha \to \pm\infty$. Assume that the load $p(x)$ is given by

$$p(x) = \sum_{k=1}^N d_k e^{\alpha_k x}, \quad x < 0, \tag{4}$$

where d_k, α_k are constant coefficients and $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_N$. Then the displacement jump admits the following series-representation

$$\chi(x) = \frac{1}{\mu}\sum_{k=1}^N d_k X_k \sum_{n=1}^\infty \frac{e^{-\sigma_n x}}{X^-(-i\sigma_n) i G'(-i\sigma_n)(\sigma_n + \alpha_k)},$$

where

$$X_k = X^+(i\alpha_k) = \exp\{\frac{\alpha_k}{\pi}\int_0^\infty \ln G(\beta)\frac{d\beta}{\beta^2 + \alpha_k^2}\}, \quad \mathrm{Im}(X_k) = 0,$$

$-i\sigma_n$ ($\sigma_n > 0$) are the elements of the countable set of roots of $G(\alpha)$ in $\mathbb{C}^-$. We note that $\mathrm{Im}\{X^\pm(\pm i\tau)\} = 0$, $\tau > 0$ and $\mathrm{Im}\{iG'(-i\sigma_n)\} = 0$. Thus, the displacement jump decays exponentially as $x \to +\infty$. It

$\psi(x) = O(e^{\lambda_0 x})$, where $\lambda_0 = \min\{\frac{\pi}{2a}, \frac{\pi}{2b}, \alpha_1\}$. Analysis of the solution in a neighbourhood of the crack tip yields that the displacement jump $\chi(x)$ is continuous and the function $\psi(x)$ is discontinuous at $x = 0$: $\chi(x) = \Lambda_0 + O(x)$, $x \to 0$; $\psi(x) = M_0 + O(x)$, $x \to -0$, $\psi(x) = 0$, $x > 0$. Here

$$\Lambda_0 = \frac{1}{\mu}\sum_{k=1}^{N} d_k(1 - X_k), \quad M_0 = \sum_{k=1}^{N} d_k X_k.$$

These formulae and the relation (2) indicate that the traction $\sigma_{zy}(x, 0)$ is bounded but discontinuous at the crack tip and its jump is

$$[\sigma_{zy}(x, 0)]_{x=-0}^{x=+0} = -\psi(-0) = -M_0.$$

The stress component $\sigma_{zx}(x, 0)$ has a logarithmic singularity at $x = 0$.

Next, we analyse the Neumann problem. Its formulation is similar to the previous problem, with the Dirichlet boundary conditions on the upper and lower parts of the strip being replaced by the homogeneous Neumann data. The unknown function $u(x, y)$ satisfies the equation and the contact conditions (1). Instead of the second relation in (1), we assume that

$$\frac{\partial u}{\partial y}(x, a) = \frac{\partial u}{\partial y}(x, -b) = 0, \quad |x| < \infty.$$

We seek the solution in the class of functions with the finite energy integral and with the following behaviour at infinity

$$|u(x, y)| < C_1, \quad x \to -\infty, \qquad |u(x, y)| < C_2 e^{-\delta x}, \quad x \to +\infty,$$

uniformly with respect to $-b < y < a$, where C_1, C_2 and δ are positive constants. To specify the solution uniquely, we impose the following orthogonality condition

$$\int_{-\infty}^{\infty} \frac{\partial u}{\partial y}(x, \pm 0)dx = 0. \tag{5}$$

The corresponding Wiener-Hopf problem becomes

$$\Phi^-(\alpha) = G(\alpha)[\mu\Phi^+(\alpha) + P^-(\alpha)], \ \alpha \in \Gamma, \quad \Gamma = \{\alpha : \mathrm{Im}(\alpha) = -\delta_0 \in (-\delta, 0)\}, \tag{6}$$

$$G(\alpha) = 1 + \frac{\mu}{\alpha}\left(\frac{\coth \alpha a}{\mu_+} + \frac{\coth \alpha b}{\mu_-}\right).$$

Here $\Phi^+(\alpha)$ is an analytic function in D^+, and the functions $\Phi^-(\alpha), P^-(\alpha)$ are analytic in D^-, where $D^+ = \{\alpha : \mathrm{Im}(\alpha) > -\delta_0\}$, $D^- = \{\alpha : \mathrm{Im}(\alpha) < -\delta_0\}$. The solution of the problem (6) is not unique:

$$\Phi^+(\alpha) = \frac{C - \Psi^+(\alpha)}{\mu(\alpha + i)X^+(\alpha)}, \quad \alpha \in D^+, \quad \Phi^-(\alpha) = \frac{C - \Psi^-(\alpha)}{(\alpha - i)X^-(\alpha)}, \quad \alpha \in D^-,$$

where C is an arbitrary constant,

$$\Psi^\pm(\alpha) = \frac{1}{2\pi i}\int_\Gamma \frac{(t + i)X^+(t)P^-(t)}{t - \alpha}dt, \ \ \alpha \in D^\pm,$$

$$X^\pm(\alpha) = \exp\left\{\frac{1}{2\pi i}\int_\Gamma \ln\left(\frac{t - i}{t + i}G(t)\right)\frac{dt}{t - \alpha}\right\}, \quad \alpha \in D^\pm.$$

However, the original Neumann boundary value problem is solvable uniquely, and constant C is determined by the condition (5) as follows $C = \Psi^+(0) - iX^+(0)P^-(0)$. Further, assuming as before that the load has the form (4) and analysing the behaviour of the solution at infinity, we get

$$\psi(x) = A + O(e^{\lambda_1 x}), \quad \chi(x) = -\frac{1}{\mu}A + O(e^{\lambda_1 x}), \quad x \to -\infty,$$

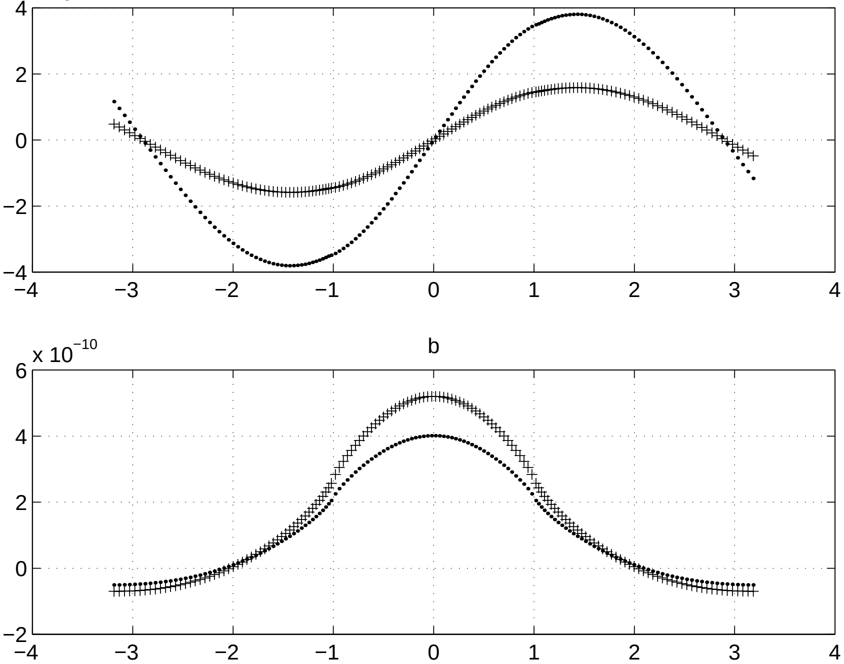

Figure 1: The displacement jumps $\chi_1(x)$ (a) and $\chi_2(x)$ (b) vs. x for the lower material brass (+ + +) and CFRP (...).

where

$$\lambda_1 = \min\ \{\alpha_1, \frac{\pi}{a}, \frac{\pi}{b}\}, \quad A = \frac{\mu_1}{X^+(0)} \sum_{k=1}^{N} \frac{d_k(\alpha_k+1)X^+(i\alpha_k)}{\alpha_k^2}.$$

Thus, in contrast to the Dirichlet problem studied in the previous section, the functions $\psi(x)$ and $\chi(x)$ do not vanish as $x \to -\infty$. For positive x, the function ψ vanish, and the displacement jump decays exponentially $\chi(x) = O(e^{-\sigma_1^{(0)}x}), \quad x \to +\infty$ ($-i\sigma_1^{(0)}$ is the first root of the function $G(\alpha)$ in D^- and $\sigma_1^{(0)} > 0$).

PLANE STRAIN PROBLEM FOR A STRIP WITH A FINITE CRACK ALONG AN IMPERFECT INTERFACE

Consider the domain with the same geometry as before but with the crack along the imperfect interface being finite. The displacement vector $\boldsymbol{u} = (u, v)$ satisfies the Lame equation

$$\nabla^2 \boldsymbol{u} + \frac{1}{1-2\nu_\pm} \nabla\nabla \cdot \boldsymbol{u} = 0.$$

The traction conditions are posed on the upper and lower parts of the boundary of the strip and on the crack faces $\tau_{xy} = f_1(x)$, $\sigma_y = f_2(x)$ as $|x| < c$, $y = 0$. The interface conditions outside the crack are

$$\tau_{xy}(x,0) = \alpha_{11}\chi_1(x) + \alpha_{12}\chi_2(x), \quad \sigma_y(x,0) = \alpha_{12}\chi_1(x) + \alpha_{22}\chi_2(x), \quad |x| > c, \ \ y = \pm 0.$$

Here $\chi_1(x) = u|_{y=+0} - u|_{y=-0}$, $\chi_2(x) = v|_{y=+0} - v|_{y=-0}$ are the displacement jumps. The constants α_{kj} are given. For an isotropic interface layer, $\alpha_{11} = \mu, \alpha_{22} = \lambda + 2\mu, \alpha_{12} = \alpha_{21} = 0$, $\epsilon\lambda, \epsilon\mu$ are the Lamé elastic moduli. We seek the solution with finite elastic energy, and assume that the displacement field decays at infinity. Introduce new functions Ψ_1, Ψ_2 such that

$$\tau_{xy}(x,0) = \alpha_{11}\chi_1(x) + \alpha_{12}\chi_2(x) + \Psi_1(x),$$

$$\sigma_y(x,0) = \alpha_{12}\chi_1(x) + \alpha_{22}\chi_2(x) + \Psi_2(x), \quad |x| < \infty,$$

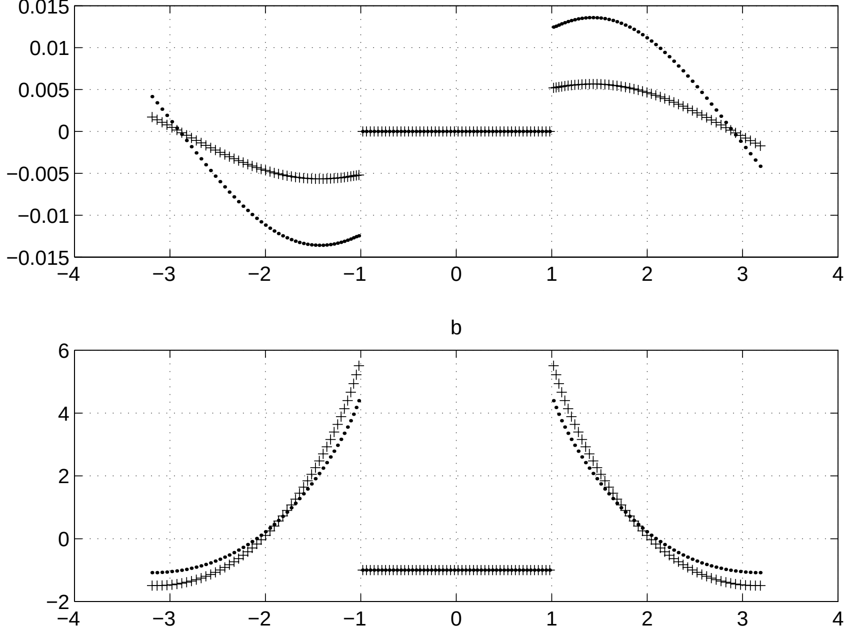

Figure 2: The stress components τ_{xy} (a) and σ_y (b) along the interface vs. x for the lower material brass (+ + +) and CFRP (...).

where $\Psi_j(x) = 0$ outside the crack. By the method of integral transforms, the original problem is reducible to the following Fredholm system of the second order

$$\Psi_k(cx) + \frac{\beta_{kj}}{\pi} \sum_{j=1}^{2} \int_{-1}^{1} \{\log|\xi - x| + K_{kj}(\xi - x)\} \Psi_j(c\xi) d\xi = f_k(cx), \quad |x| < 1, \quad k = 1,\ 2, \qquad (7)$$

where $\beta_{kj} = c\alpha_{kj}[(1-\nu_+)/\mu_+ + (1-\nu_-)/\mu_-]$, $\mu_\pm, \nu_\pm$ are the shear modulus and Poisson's ratio of the upper (lower) part of the composite strip. The function $K_{kj}(\xi - x)$ is bounded at $\xi = x$ and its first derivative has a logarithmic singularity. Analysis of the system (7) shows that the displacement jumps $\chi_1(x),\ \chi_2(x)$ are continuous at the points $x = \pm c$ and the stress components $\sigma_y(x, 0), \tau_{xy}(x, 0)$ are bounded and discontinuous at the ends $x = \pm c$. The system of integral equations (7) is solved numerically. Figures 1 and 2 show the displacement jumps and the stress components along the interface for the case $f_1 = 0, f_2 = -1, a = b = c = 1$. The upper material is aluminium with the elastic modulus $E_+ = 70\, GPa$ and Poisson's ratio $\nu_+ = 0.3$. The lower material is either $CFRP$: $E_{-(CFRP)} = 135\, GPa,\ \nu_{-(CFRP)} = 0.3$, or brass: $E_{-(Br)} = 100\, GPa,\ \nu_{-(Br)} = 0.25$. The interface layer is assumed to be made of FM 1000 and characterised by the normalised moduli, $E = 10\, GPa, \nu_0 = 0.41$ whereas the real values are given by $E_0 = 1.24\, GPa, \nu_0 = 0.41$, and thus here $\epsilon = 0.124$. The parameters that are involved in the interface condition in this case have the following values, $\alpha_{11} = 0.35\, GPa$; $\alpha_{22} = 2.3\, GPa$, and $\alpha_{12} = \alpha_{21} = 0$.

ACKNOWLEDGEMENTS

The work was supported by U.K. Engineering and Physics Sciences Research Council (EPSRC).

REFERENCES

1. Hayes, R.E. and Kolaczkowski, S.T. (1997) *Introduction to catalytic combustion.* Gordon and Breach Publishers.
2. Klarbring, A. (1991) *Int. J. Engng. Sci.* 29, 493.
3. Bigoni, D., Ortiz, M. and Needleman, A. (1997). *Int. J. Solids Structures* 34, 4305.