# Inhomogeneity effects on crack growth

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## ABSTRACT

Functionally gradient and composite materials considerably enhance our ability to tailor material properties. However, the inherent inhomogeneities pose new problems for fracture studies of these materials. This article discusses the evaluation of the effect of inhomogeneities on the crack-tip driving force in general inhomogeneous bodies and reports results for bimaterial composites. The theoretical model, based on Gurtin's configurational forces approach, makes no assumptions about the nature of the inhomogeneities or the constitutive properties of the materials. The effect of inhomogeneities on the crack-tip driving force is found by evaluating the difference between the J-integral on a contour close to the crack-tip and the J-integral on a contour in the far-field in a two-dimensional setting.

## KEYWORDS

Inhomogeneous materials, functionally gradient materials, composites, crack-driving force, J-integral.

## INTRODUCTION

Recent experiments indicate that it is possible to arrest the growth of fatigue cracks. Using bimaterial steel specimens, Suresh et al. [1] have demonstrated that cracks on the side with the lower yield stress seem to stop growing as they approach the bimaterial interface, whereas cracks in the higher yield stress side grow through the interface. Such measurements indicate that it is important to understand the influence of material inhomogeneities on crack growth. Composites and graded materials, which are invariably inhomogeneous, are being increasingly used in engineering applications, and so this issue is of relevance to applications.

One way in which inhomogeneities can influence crack-growth is by altering the nature of the crack-tip stress field. For instance, cracks parallel to a bi-material interface may either grow towards the interface or away from it [2, 3] (kindly see Kolednik [4] for a more comprehensive review). The goal of this article is to examine if there are additional ways in which the inhomogeneities can influence crack growth. We show that the energy release rate is different at the crack-tip and in the far-field, when inhomogeneities are present, and this is a second way in which inhomogeneities can influence crack growth.

## CRACK-TIP DRIVING FORCE

The model evaluates the effect of inhomogeneities by finding the difference between the J-integral on two contours - one close to the tip and another along the specimen boundary. The calculation is done in two dimensions and follows Gurtin's configurational forces approach [5, 6]. Since crack growth corresponds to the motion of the crack-tip in the reference configuration, it is best to distinguish the reference and deformed configurations. Hence results are first obtained in a finite deformation setting. Linearized versions are also reported, since the infinitesimal strain setting is widely used in fracture mechanics. For brevity, we only outline the model and refer the reader interested in a full derivation to [7].

In the configurational forces approach, two systems of forces are introduced: the classical deformational forces that act in the current configuration (such as gravity) and a new system of forces called configurational forces that act in the reference configuration. The configurational forces are responsible for kinematic changes in the reference such as the propagation of phase boundaries, thin-film growth and motion of the crack-tip $T$. Consequently there are separate balance laws for the deformational and configurational forces. Neglecting inertia and heat transfer, we consider the following mechanical setting. In the bulk we choose a deformational stress $\mathbf{S}$ (Piola-Kirchoff), a configurational stress $\mathbf{C}$ and a configurational force $\mathbf{f}$. In addition, a configurational force is taken at the crack-tip $\mathbf{f}_T$.

Consider a region $\mathcal{D}$ containing the crack-tip. The balance of deformational forces requires that

$$\int_{\partial\mathcal{D}} \mathbf{S}\mathbf{n}\, da = \mathbf{0} \qquad \text{for every region } \mathcal{D}$$

where $\mathbf{n}$ is the normal to the boundary $\partial\mathcal{D}$. Note that the only deformational force is due to the contact force from the bulk stress $\mathbf{S}$ acting on the boundary $\partial\mathcal{D}$. On the other hand, the configurational contact force, arising from the bulk stress $\mathbf{C}$ acting on $\partial\mathcal{D}$, the configurational body force as well as the configurational crack-tip force act on the region $\mathcal{D}$. Thus the balance of configurational forces requires that

$$\int_{\mathcal{D}} \mathbf{f}\, da + \int_{\partial\mathcal{D}} \mathbf{C}\mathbf{n}\, dl + \mathbf{f}_T = \mathbf{0}$$

for every region $\mathcal{D}$.

Since the bulk stresses can be singular at the crack-tip, we need to be careful in localizing these balance laws. We remove a circle $\mathcal{B}_r$ of radius $r$ centered at the crack-tip, apply the usual divergence theorem in the domain $\mathcal{D}(t)\backslash\mathcal{B}_r$ and then take the limit $r \to 0$. Please see equations (A1) and (A2) of [8] for details. One can show that the deformational and configurational force balances localize to

$$\nabla\cdot\mathbf{S} = \mathbf{0}, \qquad \nabla\cdot\mathbf{C} + \mathbf{f} = \mathbf{0} \quad \text{in the body} \tag{1}$$

$$[[\,\mathbf{S}\,]]\mathbf{p} = \mathbf{0}, \qquad [[\,\mathbf{C}\,]]\mathbf{p} = \mathbf{0} \quad \text{on the crack face} \tag{2}$$

$$\lim_{r\to 0}\int_{\partial\mathcal{B}_r} \mathbf{S}\mathbf{n}\, dl = \mathbf{0}, \qquad \lim_{r\to 0}\int_{\partial\mathcal{B}_r} \mathbf{C}\mathbf{n}\, dl + \mathbf{f}_T = \mathbf{0} \qquad \text{at the crack-tip} \tag{3}$$

Here $\mathbf{p}$ is a normal to the crack (pointing, for instance, from the bottom to the top face) and $\mathbf{n}$ is the outward normal to the boundary of the disk $\partial\mathcal{B}_r$. Equation $(1)_1$ in the bulk is the familiar equilibrium equation of continuum mechanics. Equation $(2)_1$ is trivially satisfied since the crack faces are traction free. The limiting value of the singular bulk stress vanishes at the tip $(3)_1$.

The local forms of the configurational force balance are necessary for finding the crack-tip driving force and the effect of inhomogeneities. It can be shown that the configurational stress in the bulk is nothing but the Eshelby tensor [5]

$$\mathbf{C} = \phi\mathbf{I} - \mathbf{F}^T\mathbf{S}, \tag{4}$$

where $\phi$ is the stored energy and $\mathbf{F}$ is the deformation gradient. Knowing this we can evaluate the configurational body force $\mathbf{f}$ and the configurational crack-tip force $\mathbf{f}_T$ from relations $(1)_1$ and $(3)_1$ respectively.

The dissipation $\Gamma$ is defined as the difference between the rate of working and the rate of change of the Helmholtz free energy. The crack-tip dissipation $\Gamma_T$ can be identified as the limiting value of the dissipation of the region $\mathcal{B}_r$ in the limit as the radius $r \to 0$ [9]. Based on certain conditions on the energy density $\phi$, which are satisfied by the linear elastic energy, for instance, Gurtin and Podio-Guidugli [9] show that the crack-tip dissipation is given by

$$\Gamma_T = \mathbf{v}_T \cdot \lim_{r \to 0} \int_{\partial \mathcal{B}_r} (\phi \mathbf{I} - \mathbf{F}^T \mathbf{S})\, \mathbf{m}\, dl \tag{5}$$

where $\mathbf{v}_T$ is the velocity of the crack-tip. The Clausius-Duhem version of the Second Law of Thermodynamics requires that the dissipation $\Gamma_T$ be non-negative. Then by treating the crack-tip velocity as an internal variable, we can identify the *limiting value* of the integral in (5) as the crack-tip driving force. In fracture mechanics the energy dissipated per unit crack extension is commonly used, and the crack-tip driving force can be written as

$$J_T = \mathbf{e} \cdot \lim_{r \to 0} \int_{\partial \mathcal{B}_r} (\phi \mathbf{I} - \mathbf{F}^T \mathbf{S})\, \mathbf{m}\, dl \tag{6}$$

where the unit vector $\mathbf{e} = \mathbf{v}_T / |\mathbf{v}_T|$ lies along the direction of crack growth. The above integral is the J-integral of fracture mechanics [10]. We reiterate that assumptions about the specific form for the energy $\phi$ are not necessary to derive the driving force (6), hence it is valid even for non-linear hyperelastic and elasto-plastic materials.

## EFFECTS OF INHOMOGENEITIES

Let $\mathcal{D}$ be the region between two circles - one close to the tip $\partial \mathcal{B}_r$ and another in the far-field $\partial \mathcal{B}_{far}$. Note that $\mathcal{D}$ does not include the crack or the disk $\mathcal{B}_r$. Thus only the configurational body force and contact forces act on this region, so the statement of configurational force balance for region $\mathcal{D}$ is

$$\int_{\mathcal{D}} \mathbf{f}\, da + \int_{\partial \mathcal{D}} \mathbf{C}\mathbf{m}\, dl = 0$$

The integral of the configurational contact stress can be decomposed as follows:

$$\int_{\partial \mathcal{D}} \mathbf{C}\mathbf{m}\, dl = \int_{\partial \mathcal{B}_{far}} \mathbf{C}\mathbf{m}\, dl - \int_{\partial \mathcal{B}_r} \mathbf{C}\mathbf{m}\, dl - \int_{C_D} [[\mathbf{C}]]\mathbf{p}\, dl$$

The balance law $(2)_2$ implies that $[[\mathbf{C}]]\mathbf{p} = 0$. In addition, using representation (4) and taking the limit as $r \to 0$, we get

$$\mathbf{e} \cdot \int_{\partial \mathcal{B}_{far}} (\phi \mathbf{I} - \mathbf{F}^T \mathbf{S}) \mathbf{m} dl - \mathbf{e} \cdot \lim_{r \to 0} \int_{\partial \mathcal{B}_r} (\phi \mathbf{I} - \mathbf{F}^T \mathbf{S}) \mathbf{m} dl = -\mathbf{e} \cdot \int_{\mathcal{D}} \mathbf{f} da\ . \tag{7}$$

Inhomogeneities can be modeled by making the free energy $\phi$ depend on the reference coordinate $\mathbf{x}$, and we take $\phi = \phi(\mathbf{F}, \mathbf{x})$. Note that the balance law $(1)_2$ and representation (4) imply that

$$\mathbf{f} = -\nabla \cdot (\phi \mathbf{I} - \mathbf{F}^T \mathbf{S})$$

Now using (i) $\mathbf{S} = \partial \phi / \partial \mathbf{F}$, (ii) the equilibrium condition $(1)_1$ and (iii) the fact that the order of partial differentiation can be interchanged, one can show that $\mathbf{f} = -\nabla_x \phi(\mathbf{F}, \mathbf{x})$. Thus, identifying the first term in (7) as the far-field J-integral, we get

$$J_{far} - J_T = \mathbf{e} \cdot \int_{\mathcal{D}} \nabla_x \phi(\mathbf{F}, \mathbf{x}) da \tag{8}$$

For homogeneous hyperelastic materials $\phi = \phi(\mathbf{F})$, and equation (7) would imply the path-independence of the J-integral. A similar expression can be obtained when the deformation is infinitesimal, in which case (4) is replaced by $\mathbf{C} = \phi\mathbf{I} - \nabla\mathbf{u}^T\mathbf{T}$ where $\nabla\mathbf{u}$ is the displacement gradient and $\mathbf{T}$ is the Cauchy stress.

**Elastic bimaterial composite**. Now consider a composite made of two materials with different stiffness but the same Poisson's ratio. The linear elastic strain energy can be written as $\phi(\lambda, \nu, \mathbf{E}) = \lambda\psi(\nu, \mathbf{E})$ (e.g. see pg. 246 of Timoshenko [11]) where $\lambda$ denotes the stiffness, $\nu$ the Poisson's ratio and the function $\psi \geq 0$ is independent of the stiffness. Thus $\nabla\phi = \psi\nabla\lambda$, and we can rewrite (8) as

$$J_{far} - J_T = \mathbf{e} \cdot \int_{s_1}^{s_2} \left[\psi \int \nabla\lambda \, dn\right] ds$$

where we parametrize the interface with the arc-length $s$ and write the element of area $da = dsdn$ with $n$ denoting the coordinate in the direction normal to the interface. Since the gradient is maximum in the direction normal to the interface, we obtain

$$J_{far} - J_T = (\lambda^+ - \lambda^-) \int_{s_1}^{s_2} (\mathbf{e} \cdot \mathbf{n}) \, \psi(\mathbf{E}) ds \tag{9}$$

where $\lambda^\pm$ correspond to the material on the two sides of the interface and the interface normal points from the $-$ to the $+$ side. [Visualizing the interface as being smeared over a distance $\Delta n$ helps in obtaining (9)]. The limits of integration $s_1$ and $s_2$ correspond either to geometrical features of the specimen or can be fixed from the accuracy required for solutions in numerical calculations.

The effect of the jump in the stiffness at the bimaterial interface is given by relation (9). First, recollect that $\psi$ is just the stored elastic energy divided by the stiffness, so it is non-negative. Hence the inhomogeneity will either aid or inhibit crack growth depending on the sign of $(\lambda^+ - \lambda^-)$; positive sign inhibits growth, while negative sign aids growth. Second, the inhomogeneity effects appear only when the bimaterial interface is close to the crack-tip. More precisely, according to the linear elastic K-field, $\psi$ and the integral scale as $1/r$ where $r$ is the distance from the crack-tip.

If the interface is flat and the crack is initially parallel to the interface, then $\mathbf{e} \cdot \mathbf{n} = 0$, and there is no contribution from (9). Notice that the integral in (5) is maximum in the direction $\mathbf{e}$, and this corresponds to the maximum value of $J_T$. If the presence of the interface causes the maximum dissipation to be in a direction different from the initial crack direction, then we may expect the crack to turn in that direction. Following this, the interface will influence crack growth through (9).

For a crack perpendicular to the bimaterial interface, $\mathbf{e} \cdot \mathbf{n} = 1$. If the crack is on the side with smaller stiffness ($\lambda^+ > \lambda^-$ so $J_{far} > J_T$), then the interface inhibits the crack growth and vice-versa. We now obtain a simple estimate for the integral in (9) by assuming that the usual K-field of LEFM can be used to evaluate the elastic energy. For plane strain [11]

$$\psi(\mathbf{E}, \nu) = \frac{1}{2(1+\nu)} \left[\frac{1-\nu}{1-2\nu}(\epsilon_x^2 + \epsilon_y^2) + \frac{2\nu}{1-2\nu}\epsilon_x\epsilon_y + \frac{1}{2}\gamma_{xy}^2\right]$$

where $\epsilon_x, \epsilon_y$ are axial strain components and $\gamma_{xy}$ is the engineering shear strain.

A stress formulation governs the crack-tip equilibrium problem, and solutions are obtained in terms of stresses, not strains. Hence there is a need to write $\psi$ in terms of stresses. Remember the derivative $\mathbf{f} = -\nabla_x\phi(\mathbf{E}, \mathbf{x})$ is taken at fixed strain, but after taking the derivative we can change variables and replace strain with stress. Using the elastic stress-strain relations for plane strain, we obtain

$$\psi = \frac{1+\nu}{2\langle\lambda\rangle^2} \left[(1-\nu)(\sigma_x^2 + \sigma_y^2) - 2\nu\sigma_x\sigma_y + 2\sigma_{xy}^2\right]$$

where $\sigma$ denotes a linear elastic stress component; however, the Young's modulus is different on either side of the interface, and hence we take the average value ($\langle\lambda\rangle = (\lambda^- + \lambda^+)/2$). Substituting the expressions for the usual mode-I crack-tip K-field stresses we get

$$\psi = \frac{1+\nu}{2\langle\lambda\rangle^2} \left(\frac{K^2}{4\pi}\right) \left[2 - 2\nu - \cos^2(\theta/2)\right] \cos^2(\theta/2)$$

where $K$ is the stress intensity factor, $r$ and $\theta$ are polar coordinates, and the origin is at the crack-tip.

Suppose that the bimaterial interface is at a distance of $l$ in front of the crack-tip. Then $cos(\theta) = l/r$ and we can write

$$\psi = \frac{(1+\nu)K^2}{8\pi\langle\lambda\rangle^2}\left[\frac{3-4\nu}{r} + \frac{(2-4\nu)l}{r^2} - \frac{l^2}{r^3}\right]$$

The arc-length is nothing but the (vertical) y-coordinate and noting that $r = \sqrt{l^2+y^2}$, we can evaluate the integral in (9) as

$$\int_{-b}^{b} \psi\, dy = \frac{(1+\nu)K^2}{8\pi\langle\lambda\rangle^2}\left[(3-4\nu)ln\left(\frac{\sqrt{1+\alpha^2}+\alpha}{\sqrt{1+\alpha^2}-\alpha}\right) + 4(1-2\nu)tan^{-1}(\alpha) - \frac{2\alpha}{\sqrt{1+\alpha^2}}\right] \quad (10)$$

where $\alpha = b/l$ and the limit of integration $b$ either corresponds to the physical dimensions of the specimen or can be chosen depending on the accuracy required for results. For a given value of $\alpha$ the effect of inhomogeneities can be evaluated using (9, 10).

If we treat $b$ as a fixed parameter, then variations in $\alpha$ are due to variations in $l$. Figure 1 shows $\Psi$ vs. $\alpha$ where

$$\Psi = \left[\int_{-b}^{b} \psi\, dy\right]\left(\frac{8\pi\langle\lambda\rangle^2}{(1+\nu)K^2}\right)$$

The inhomogeneity effect $\Psi$ does not reach an asymptotic value for large values of $\alpha$, but instead $\Psi \to \infty$ as $\alpha \to \infty$. This means that the inhomogeneity effect $\Psi$ becomes very large as the distance between the crack-tip and the interface becomes vanishingly small. Here we have assumed that the stress-intensity factor $K$ remains constant; however, other studies have found that $K$ changes with $\alpha$, and hence Figure 1 should be interpreted with this caveat.

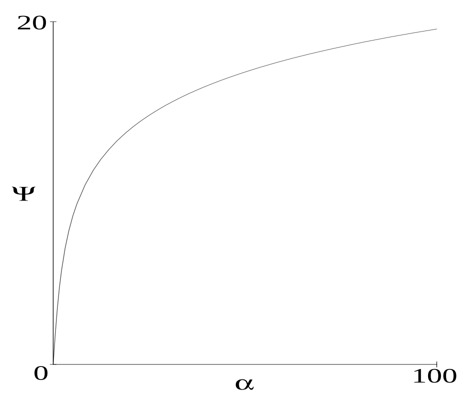


Figure 1: $\Psi$ vs. $\alpha$

## CONCLUSIONS.

1. It is well-known that inhomogeneities can alter the crack-tip stress field. This paper shows that inhomogeneities can influence crack growth in a second way - by causing a difference in the crack-tip and far-field energy release rate [equation (8)].
2. For a bimaterial composite, a crack in the less stiff material is inhibited as it approaches the interface, whereas the growth of a crack in the stiffer material is aided by the interface.

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