FRACTURE CRITERION FOR PIEZOELECTRIC MATERIALS WITH DEFECTS BASED ON ENERGY DENSITY THEORY

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ABSTRACT

A failure criterion, which was established based on the energy density factor, is extended for an elliptical cavity or a line crack embedded in an infinite piezoelectric solids, subjected to a combined in-plane electrical and mechanical loading. In the present analysis, the exact electric boundary conditions are applied at the rim of the cavity/crack. The direction of crack initiation or subsequent post-failure, and the critical loads for fracture, can be predicted using the total energy density factor, S. This factor is a function of the aspect ratio of the elliptical cavity, the electromechanical loading, core region outside the crack tip, permittivity of vacuum and material constants. The results obtained agree with the experimental observation, i.e. a positive electric field enhances crack growth while a negative electric field impedes crack growth. Moreover, the results indicate that the critical fracture loads are under-estimated by the assumption of impermeable crack and over-estimated when the crack is assumed to be permeable for $E_z^{app}>0$, where E_z^{app} is the applied electric field. However, the fracture loads are over-estimated by the assumption of impermeable crack and under-estimated when the crack is assumed to be permeable for $E_z^{app}<0$. The energy density criterion has the advantage of possessing the capability to implement the exact electric boundary conditions. This is due to the fact that the criterion can link the behavior of a crack to that of an elliptical cavity by consistent application of this criterion to a thin layer near the cavity/crack boundary.

KEYWORDS

Energy density factor, piezoelectric material, fracture criterion, energy release rate, elliptical cavity, crack

INTRODUCTION

The fracture problems of piezoelectric materials have received much attention in the last ten years. Thus, vast amount of theoretical results have been obtained by many researchers [1-8]. However, nearly all the previous analyses were based on the assumption of impermeable or permeable crack. In fact, these two extreme cases are not realistic. Since the dielectric constant of the air or vacuum inside the crack is neither zero nor the dielectric permittivity of the material [3, 7], the crack problems in a piezoelectric material should be treated as electric inclusion problems. Consequently, Sosa and Khutoryansky [9] used the series expansion method to address the plane problem of a transversely isotropic piezoelectric medium with an elliptical hole.

Many fracture criteria for piezoelectric materials have been proposed in the recent years, e.g., the total potential [4,6], the mechanical [8] and the local [10] energy release rates, and the energy criterion considering domain switching dissipation [11]. Pak [4] found that the presence of applied electric field always reduced the total potential energy release rate. This implies that a crack is impeded by the electric field regardless of its direction. This conclusion contradicts the existing experimental data. Later, Park and Sun [8] proposed to use only the mechanical part of the energy as a criterion by arguing that fracture is a mechanical process and, therefore, it should be controlled by the mechanical part of the energy release rate. But, this argument is unsound because there is no fundamental reason to separate a physical process into an electric part and a mechanical part, in view of the fact that all mechanical forces are of electromagnetic origin. As a result, Gao et al. [10] proposed a criterion based on the local energy release rate of an electrically yielded crack. However, the effect of domain switching cannot be accounted for using this model. In line with Griffith's theory on mechanical fracture, Fang et al. [11] proposed a criterion based on energy balance with the consideration of domain switching dissipation. However, they only proposed an energy balance approach and the results are the same as those of Gao et al [10].

Recently, Shen and Nishioka [12] used strain energy density theory to develop a fracture criterion for piezoelectric materials. Their theoretical result agrees qualitatively with the empirical evidence by assuming impermeable crack. In the present study, the energy density fracture analysis of an elliptic cavity or a crack, in a transversely isotropic piezoelectric solid subjected to remote loading, is carried out. This study adopts the approach and the assumption of exact electric boundary conditions employed by Sosa and Khutoryansky [9]. First of all, a surface layer criterion [13] is used to locate the position where the fracture of the elliptic cavity is expected to initiate. The direction of crack initiation and subsequent post-failure can be described by the total energy density factor, S, from which the critical loads for fracture can be predicted.

SOLUTION FOR AN ELLIPTIC CAVITY

Consider a central elliptic cavity of major and minor semi-axes, *a* and *b*, respectively, embedded in an infinite poled piezoelectric ceramic, which is subjected to a remote in-plane electrical loading, E_z^∞, and mode I mechanical loading, σ_{zz}^∞ as shown in Figure 1. The cavity is assumed to be filled with a dielectric medium of permittivity, ε_0, and it is free of traction force and charge. The ceramic is poled in the positive direction of the z axis.

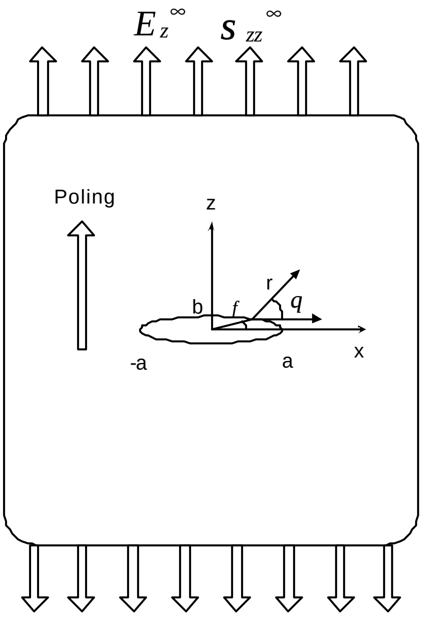

Figure 1: Schematic diagram of an elliptical cavity embedded in a piezoelectric plate

By applying the procedure proposed by Sosa and Khutoryansky [9], the field solutions for two-dimensional problems based on the exact electric boundary conditions can be expressed as

$$\sigma = \begin{pmatrix} \sigma_{xx} \\ \sigma_{zz} \\ \sigma_{xz} \end{pmatrix} = \begin{pmatrix} \sigma_{xx}^{\infty} \\ \sigma_{zz}^{\infty} \\ \sigma_{xz}^{\infty} \end{pmatrix} + 2\,\mathrm{Re}\sum_{k=1}^{3} \begin{Bmatrix} \mu_k^{2} \\ 1 \\ -\mu_k \end{Bmatrix} L_k\,; \qquad D = \begin{pmatrix} D_x \\ D_z \end{pmatrix} = \begin{pmatrix} D_x^{\infty} \\ D_z^{\infty} \end{pmatrix} + 2\,\mathrm{Re}\sum_{k=1}^{3} \begin{Bmatrix} \mu_k \\ -1 \end{Bmatrix} \lambda_k L_k\,;$$

$$\varepsilon = \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{zz} \\ 2\varepsilon_{xz} \end{pmatrix} = \begin{pmatrix} \varepsilon_{xx}^{\infty} \\ \varepsilon_{zz}^{\infty} \\ \varepsilon_{xz}^{\infty} \end{pmatrix} + 2\,\mathrm{Re}\sum_{k=1}^{3} \begin{pmatrix} a_{11}\mu_k^{2} + a_{12} - b_{21}\lambda_k \\ a_{12}\mu_k^{2} + a_{22} - b_{22}\lambda_k \\ -a_{33}\mu_k + b_{13}\lambda_k\mu_k \end{pmatrix} L_k \quad ; E = \begin{pmatrix} E_x \\ E_z \end{pmatrix} = \begin{pmatrix} E_x^{\infty} \\ E_z^{\infty} \end{pmatrix} + 2\,\mathrm{Re}\sum_{k=1}^{3} \begin{pmatrix} 1 \\ \mu_k \end{pmatrix} \kappa_k L_k \qquad (1)$$

where Re denotes the real part of a complex valued quantity; complex coefficients, μ_k, λ_k, κ_k, L_k, which are functions of material constants are given in Soh et al [14].

For an elliptical notch, a surface layer criterion proposed by Sih [13] can be employed as a preliminary judgment to locate the position ϕ, at which fracture may initiate from the notch surface. Note that ϕ denotes the angle between the direction of crack propagation and that of the positive major semi-axis of the elliptical notch. Once the angle ϕ is determined, the knowledge of the energy stored in an element outside the core region (near the apex of the elliptical notch) is used as a means for establishing the failure location at some points in the bulk of the solid.

The energy associated with the surface layer can be derived from the mechanics of thin layer. On Γ(surface of cavity), the radial stress is equal to zero. Thus, the element in the thin layer is subjected to uniaxial tension. Therefore, the strain energy per unit area of the surface layer can be expressed as

$$g_e = \left(\frac{1}{2} \boldsymbol{s}_{ss} \boldsymbol{e}_{ss} + \frac{1}{2} E_s D_s \right) \boldsymbol{d} \tag{2}$$

where $\boldsymbol{d}$ is a parameter for quantifying the surface condition, and σ_{ss}, ε_{ss}, D_s, E_s are given in Soh et al. [14]. Note that $\gamma_e/\delta\sigma^2\pi a$ is a convenient non-dimensional form for indicating strain energy. A notched specimen may be loaded to failure and the strain energy per unit surface layer can be computed from equation (2). The location of initial failure (i.e., the angle of crack propagation, $\boldsymbol{f}$) on the notch boundary is then determined by setting $¶g_e/¶\boldsymbol{f} = 0$ for g_e to reach its maximum value.

ENERGY DENSITY CONCEPT

For a piezoelectric material containing an elliptical cavity, the surface layer energy criterion is only applicable to the surface of the cavity. The strain energy density criterion is then employed to predict the trajectory of crack propagation or failure path at the interior points near the surface. Similar to the pure mechanical case [13], the energy density function for the electromechanical case in the element *dV* of a general three-dimensional system can be written as

$$\frac{dW}{dV} = \frac{1}{2} \boldsymbol{s}_{ij} \boldsymbol{e}_{ij} + \frac{1}{2} E_i D_i \tag{3}$$

and the strain energy density field, S, is expressed as

$$S = r \frac{dW}{dV} \tag{4}$$

In order to obtain better accuracy, the complete strain-energy density expression is used in the present study rather than that with only singular terms. The fundamental parameter in this theory, *S*, is direction sensitive in the sense that it predicts the direction of crack propagation. This is accomplished by calculating the stationary value of *S* or *dW/dV*, with *r* being the radial distance measured from the crack front. The minimum *S* value, i.e., $S_{\min}$, is related to dilatation of material elements and is associated with the creation of a free surface that fracture is expected along the line of crack extension.

For the present two-dimensional problems, the direction of crack propagation can be determined by a single variable $\boldsymbol{q}$ with different values of r giving the same trend. Hence, the necessary and sufficient conditions for the strain energy density factor *S* to be minimized are as follows:

$$\frac{\partial S}{\partial \boldsymbol{q}} = 0 \qquad \& \qquad \frac{\partial^2 S}{\partial \boldsymbol{q}^2} > 0 \qquad at\, \boldsymbol{q} = \boldsymbol{q}_0 \tag{5}$$

The minimum energy density factor, which occurs at the crack initiation direction, $\boldsymbol{q}_0$, can be determined from equation (5).

DISCUSSIONS

In order to demonstrate the suitability of the energy density theory for solving the problem of piezoelectric failure in the case of exact electric boundary conditions, an elliptical crack embedded in a piezoelectric material, PZT-4, is considered. For an infinite plate containing a central elliptical cavity, subjected to remote electro-mechanical loading, failure will occur at around $\phi=-9^0$ for b/a=0.5, around $\phi=-5^0$ for b/a=0.25, and around $\phi=-3^0$ for b/a=0.125 for three different load ratios ($E_z^{\infty}/\sigma_{zz}^{\infty}$=-0.01,0,0.01). Moreover, for the same remote loading, no matter in which direction the electric field is applied, increase of loading is allowed when the notch tip becomes more blunt. This coincides with the conventional fracture criterion.

Naturally, one would like to know how the crack would extend from the initial failure location on the surface of the elliptical notch. As mentioned above, in order to determine the direction of crack propagation, $\theta = \theta_0$, the stationary values (S_{min}) of the strain energy density for various radius vectors need to be determined. The calculated results show that for different radius vectors, directions of the applied electric field, and permittivity of the medium inside the elliptical cavity, the direction of subsequent fracture is always horizontally outward, i.e., in $\theta = 0$ direction.

Figure 2 shows the effects of the applied electric field on the fracture loads for a crack with half-length a=0.01m for the case of exact, impermeable and permeable boundary conditions, i.e., $\varepsilon_0 = 8.85\times 10^{-12}\, NV^{-2}$, $\varepsilon_0 = 0.0 NV^{-2}$ and $\varepsilon_0 = 5 \times 10^{-9}\, NV^{-2}$, respectively. The critical load of this specimen is 2 MPa when no electric field is applied. It should be noted that b is made approaching zero instead of equal to zero (the crack keeps open). The energy release rate criterion is shown in the same figure for comparison.

It is interesting to note that the effects of the direction of the applied electric field on crack propagation for the case of sharp crack subjected to different electric boundary conditions have the same trend. That is a negative electric field impedes crack propagation while a positive electric field enhances crack propagation. Moreover, the fracture loads are found to be under-estimated by the assumption of impermeable crack and over-estimated when the crack is assumed to be permeable for $E_z^{app}>0$. However, the fracture loads are over-estimated by the assumption of impermeable crack and under-estimated when the crack is assumed to be permeable for $E_z^{app}<0$. In contrast to the energy density criterion, the fracture loads calculated based on the criterion of energy release rate are independent of the direction of the applied electric field, which contradicts many experimental observations. It is physically incorrect for the fracture load to increase with increasing E_z^{app} increases because it is impossible that the load carrying capacity of the cracked specimen increases as the applied load is increased. All the critical stresses must decrease as the load is increased. Note that the four curves intersect at the point of zero applied electric field, which is the case of pure mechanical loading

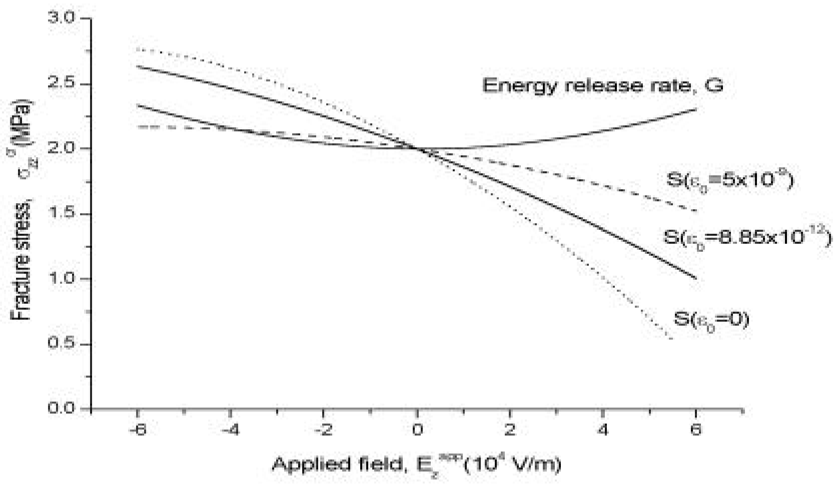

Figure 2: Effect of the applied electric field on the critical stress of a line crack

CONCLUSIONS

The objective of this study is to extend a failure criterion, based on the strain energy density theory, for an elliptical cavity or a line crack embedded in an infinite piezoelectric solid, subjected to a combined in-plane electrical and mechanical loading. This has essentially been accomplished through the consistent application of the strain energy density criterion to a thin layer near the cavity/crack boundary and material element in the interior region of the solid. The total energy density factor, *S*, depends on the material constants, b, a, polar angle, core region near the apex of the elliptical cavity, and the permittivity of vacuum. The core region on the cavity/crack boundary remains unspecified in size because of different material behavior external to the core region. Unstable crack extension is assumed to occur when some small element just outside the core region has absorbed as much elastic energy as possible and releases it to allow material separation.

As discussed above, the location on the free surface of a cavity, at which the surface layer energy is the maximum, may be postulated to be the location where initial failure occurs. Subsequently, the immediate post-failure crack direction may be determined by minimizing the total energy density factor. The results obtained in this study by introducing the exact electric boundary conditions on the cavity/crack boundary are more realistic because it reflects the practical situation. In short, the superiority of the energy density theory has been clearly shown from its capability of adopting the exact boundary conditions.

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