A NEW APPROACH FOR IDENTIFYING THE HEAT SOURCES IN TWO DIMENSIONAL POTENTIAL PROBLEMS USING THE DISCRETE INTEGRAL METHOD

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ABSTRACT

Analysis of inverse problems has already been performed in various fields. In many cases, it seems that a priori assumption for the solution are needed. This requirement causes a contradiction on the analysis.

On the other hand, we have developed the discrete integral method(DIM) utilizing the delta function. We have noticed that the DIM will become one of the excellent schemes to solve the inverse problem since it can naturally express the function of solution regardless of the type of function and can solve it without any assumptions. In this paper, we attempt to apply the DIM to the two-dimensional inverse problems. As one of the examples of two-dimensional inverse problems, we demonstrate a solving scheme for identifying the distribution of heat sources in the heat conduction problem.

Through some numerical verification, it is proved that the present scheme gives accurate and natural solutions without presumptions.

KEYWORDS

Boundary element method, Discrete integral method, Inverse problems, Potential problem

INTRODUCTION

One of the defaults in the analysis of inverse problem[1,2] is that usually a priori assumptions concerning the functional shape, the number of functions, definitive area of the function, and so on for the solution are required. As one of the consecutive studies to conquer this subject, this study is intended to establish and to suggest one of new solution schemes in the analysis of the inverse problems. We begin to introduce a way of functional approximation using the delta function. We develop the way to a new integral scheme named as the discrete integral method (DIM). In using the DIM, a domain integral can be converted to the sum of a boundary integral and values at some selected points utilizing the property of the delta function. We attempt to apply them to the identification of physical quantities expressed by an integration.

We have already established an analysis scheme for one dimensional inverse problem using this scheme[3, 4]. In this study, we try to expand this scheme to two dimensional cases. Here, the scheme for identifying heat source distribution in the heat conduction problem are demonstrated as an instance. The heat source distribution is expressed as an inhomogeneous term in the governing differential equation, and is formulated as a domain integral term in the boundary element method (BEM). Therefore, it is convenient to combine the present identification scheme with the BEM.

1. INSTITUTION OF TWO DIMENSIONAL INVERSE PROBLEM

1.1 *Integral equation of heat conduction problem by BEM*

For the steady heat conduction problem with internal heat sources, the integral equation which gives temperature on a point p can be written as[7,8]:

$$u(p) = \int_{\Gamma} \left[q^* (p,Q) \{ u(Q) - u(P) \} - u^* (p,Q) q(Q) \right] d\Gamma(Q) + \int_{\Omega} b(Q) u^* (p,Q) \, d\Omega(Q) + u(P) \qquad (1)$$

Here, u is the temperature, q is the flux, b is a function which expresses the distribution of heat source, P is a point on the boundary nearest to an internal point p, Q is a moving point within the range of integral, Γ indicates the boundary of the domain and Ω indicates the whole domain. u^*, q^* are fundamental solutions about temperature and heat flux, respectively, which are written as

$$u^* (p,Q) = -\frac{1}{2\pi} \ln r, \qquad q^* (p,Q) = \frac{\partial u^*(p,Q)}{\partial n} \qquad (2)$$

where r expresses the distance between two points of p and Q, n expresses outward normal on boundary Γ.

Inverse problem instituted in this paper is to estimate the heat sources distribution b in the second term on the right side of Eqn.1 using the information of the boundary condition and temperature distribution.

1.2 *Two dimensional discrete integral method utilizing the delta function*

First, we introduce a tool, named the discrete integral method(DIM), to solve the two-dimensional inverse problems. In the present problem, the domain integral term in Eqn.1. becomes the target.

Usually the heat source distribution function b included in that integral term is approximated by the isoparametric quadratic element, then the integration is performed by using the Gauss numerical integration. Or this term is transformed to boundary integration by using multiplex reciprocity method[5,6]. On the other hand, instead of these methods, the approximating way using Dirac's delta function can be developed. Namely, the function is approximated as follows:

$$\nabla^k b = \sum_{i=1}^{m} B_i \delta(x - x_i) \qquad (3)$$

Here, ∇^k is the k-th Nabla differential operator, B_i is the strength of the delta function, x_i is its applied position, and m is their number.

Further, we multiply a new function Z^* defined as

$$\nabla^k Z^* = u^* (p,Q), \quad \text{namely,} \quad Z^* = \frac{r^4}{256\pi}(3 - 2\ln r) \qquad (4)$$

with the both sides of Eqn.3 and integrate them over the whole domain. Then, the following equation can be derived by integration by part when $k = 2$:

$$\int_{\Omega} b u^* d\Omega = \int_{\Omega} b \nabla^2 Z^* d\Omega = \int_{\Gamma} (b \nabla Z^* - \nabla b Z^*) \mathbf{n} d\Gamma + \sum_{i=1}^{m} Z^* (x_i) B_i \qquad (5)$$

Eqn.5 shows that a given domain integration term can be calculated by using boundary integration and the strength B_i (which is unknown quantity) of concentrated source on discrete points. We call this series of schemes as the discrete integral method, and try to introduce it to identification of heat source distribution.

1.3 *Application to inverse problem*

The strength B_i in Eqn.3 are unknown quantities. In order to determine their values, we use the information of temperature distribution. By substituting Eqn.5 into Eqn.1, we obtain the following equation:

$$u(p)-u(P)=\int_\Gamma\Big[q^*\{u(Q)-u(P)\}-u^*q(Q)\Big]d\Gamma(Q)+\int_\Gamma(b\nabla Z^*-\nabla bZ^*)\mathbf{n}d\Gamma+\sum_{i=1}^{m}Z^*(x_i)B_i \tag{6}$$

Here, by monitoring $u(p)$ on m-points, we can compose a set of simultaneous equations with the dimension of m. The differential terms of b, exist in boundary integration term on right side of Eqn.6 can be eliminated by the interpolating as shown later.

Then, by using B_i obtained in this way, it becomes possible to interpolate the function b on any point s, and the distribution of heat generation is obtained as follows. Namely, we multiply a function Y^* defined as

$$\nabla^k Y^* = -\delta(x-p), \qquad \text{namely,} \qquad Y^*=\frac{r^2}{8\pi}(1-\ln r) \tag{7}$$

with both sides of Eqn.3, and integrate them over the whole domain, then, it becomes

$$\int_\Omega \nabla^k bY^* d\Omega = \int_\Omega \sum_{i=1}^{m} B_i\delta(x-x_i)Y^*d\Omega \tag{8}$$

Eqn.8 is transformed by means of integration by part:

$$b(p)=\int_\Gamma(\nabla bY^*-b\nabla Y^*)\mathbf{n}d\Gamma-\sum_{i=1}^{m}B_iY^*(x_i,p) \tag{9}$$

Eqn.9 gives the distribution of heat sources. Thus, we can identify the existence of the heat source distribution, its type as well as its number without any a priori assumptions.

2. EXAMPLES

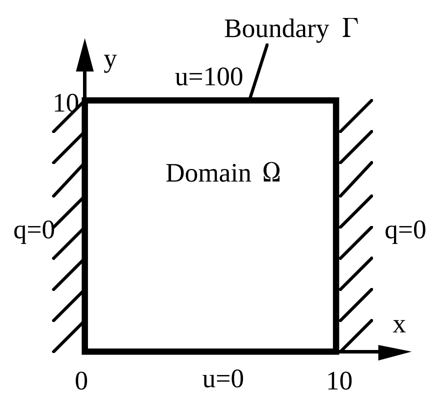

Figure 1: Analysis model and boundary condition

Here shows the practical analysis of the inverse problem by using the scheme mentioned above. In all cases, heat source distribution inside the domain is identified from the given data of boundary conditions and temperature distribution in the domain. Analysis model is a 10×10 square plate, and the boundary conditions are as shown in Figure 1. Furthermore, m, the number of points positioned as point sources is fundamentally 9 in both x and y direction, so the total number is 81. They are arranged at even intervals in the domain excluding the boundary.

2.1 *Examples for distributed heat source*

First, as an analysis example for distributed heat source, the case of a heat source distribution expressed by $b = -x + y + 13$ applies on the whole domain, namely, on the range of $0 \leq x \leq 10, 0 \leq y \leq 10$ is examined. The result is shown in Figure 2. In this figure (including all of following figures), x-axis and y-axis correspond to those of Figure 1 respectively, and the axis of vertical direction expresses heat source distribution b which is identified by Eqn.9.

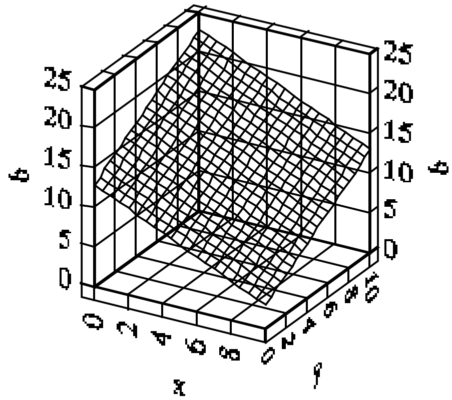

Figure 2: Identification of heat source given by $b = -x + y + 13$

Figure 2 shows that the identified data reappears accurately the given distribution. They agree in 6 decimal places or more. Because it is a case that the distributed shape of heat source becomes 0 by 2nd differential operation, very accurate result can be obtained with $k = 2$ in Eqn.3.

Figure 3 shows the result for a uniform heat source with the magnitude 5 distributes in a rectangular region of $4 \leq x \leq 6, 5 \leq y \leq 7$. The equation with the order of $k = 2$ is not available for this problem because of too big errors, so we use that of $k = 4$ from this problem. From the figure, we can clearly distinguish the heat source distribution in the right region although not any presumption is used. However, it is noticed that a little overshoot appears at the edge of the step. This shows the limit of the scheme and, you may also noticed the unevenness of the distribution of heat source. The amount of the overshoot is about 10% at most. If these identified results are considered to be insufficient, the denser arrange of the points can be employed, or the secondary identification based on the primary identified data can be performed as it will be mentioned hereafter. Besides, the shape of step part in this figure has a little trapezoidal, it depends on the intervals of the point source. If the number of points is increased, the sharper edge will appear.

Instead of a rectangular region, if the heat source has a circular region, almost the same behavior of the

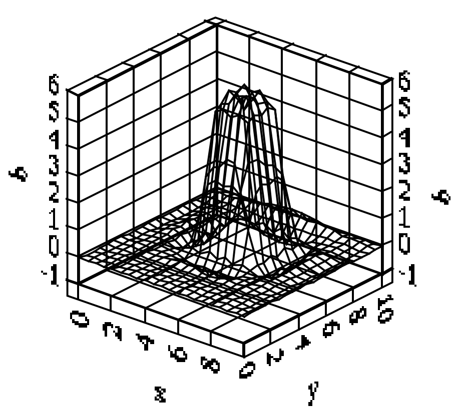

Figure 3: Identification of localized heat source

identified result can be observed.

2.2 ***The secondary identification for distributed heat source***

If the accuracy obtained by the scheme as mentioned above is not enough, it can be improved by using data of the primary identified result. Here, we will discuss the case of the further identification based on the result of Figure 3. From the result in the primary step, we can suppose the region where there is no heat source. Therefore, we try to reidentify the heat source by rearranging the point source on the part where the value from the first calculation is not zero.

From Figure 3 we can presume that a rectangular of $4 \leq x \leq 6, 5 \leq y \leq 7$ is the region where heat source may exist. Therefore, we arrange the point source only in this region, and perform the re-identification (the secondary identification). The results are improved with extremely high accuracy which error is 0.1% or less.

2.3 ***Examples for concentrated heat source and its secondary identification***

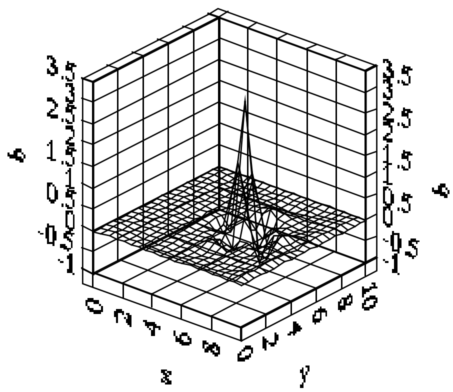

Figure 4: Identification of a concentrated heat source

A concentrated heat source with magnitude 2 applies on the position of $x = 6, y = 5$ is instituted. Identified result for this problem is shown in Figure 4. It can be seen in the figure that the distribution has a sharp peak round $x = 6, y = 5$, so we can consider a concentrated heat source exists there. However, the value of peak is not 2 which was instituted, and its applied region is very indistinct. Such an abrupt change caused because the concentrated heat source is attempted to be expressed in the form of distribution. Therefore, it is impossible essentially for this analysis scheme to distinguish a concentrated source from a distributed one in a very small region by once identification. Nevertheless, it seems that this is not a so serious problem practically. Essentially the distinction is not necessary, and as will be shown later, reasonable value of its magnitude and position also can be obtained.

To obtain the magnitude of the distribution b which is possibly a concentrated heat source as it has a sharp peak, we can integrate it over a circular region with any radius. It is calculated by the following equation which is a direct integration of Eqn.9:

$$F = \int_0^R \int_0^{2\pi} b(p)\rho d\theta d\rho \tag{10}$$

where F is the identified value regarded as the magnitude of concentrated heat source, ρ is the radius of the peak with the center point of p_0, R is the radius of integral region and θ is the angle around p_0. The result is shown at the left side of table 1. It can be seen though the error is large in abrupt changing part when integral region is small, very good value is obtained when the radius R becomes larger.

When the accuracy in Table 1 is not enough, we can perform the secondary identification by using data of the primary identification, as done in the case of distributed heat source, so that it can be distinguished from a distributed heat source more clearly. Further, as a concentrated heat source, its position and magnitude

Table 1: Identified value of the source strength in Fig.4

1st Identification		2nd Identification	
Integral Radius	Result	Integral Radius	Result
0.5	1.6034	0.05	2.1535
1.0	2.6018	0.1	3.2735
2.0	1.1992	0.2	1.0837
3.0	2.0016	0.3	1.9991

can be identified definitely. From the result in Figure 4, the point source is positioned in the region of only $5.5 \leq x \leq 6.5, 4.5 \leq y \leq 5.5$ which is a smaller region, and re-identification (the secondary identification) is performed. We found that the distribution concentrates in a much smaller region than the primary identification, and the value of its peak also becomes about 150 times larger. The same as the primary identification, the computed value for magnitude of concentrated heat source by using Eqn.10 is shown at the right side of table 1. Comparing with the result at the left, it can be seen that extremely accurate identified value is obtained in the region with much smaller radius. On the other hand, now the region where the data changes abruptly at the primary identification becomes 0, and the part with abrupt change is limited in a very narrow region. This behavior can not be considered as a distributed one, so it can be identified sufficiently that it is a concentrated heat source. We think the further study is indispensable, however, it is expected that the identification with sufficient accuracy will be possible by devising how to do the re-identification well.

3. CONCLUSIONS

The discrete integral method utilizing the delta function was introduced, and a new analysis scheme for two dimensional inverse problem based on the discrete integral method was suggested. From the analysis examples, it was shown that the heat source could be identified naturally and accurately without any a priori assumptions such as for the kind of applied heat source, its distributed shape, the number, and so on.

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