# FINITE DEFORMATION ENERGY RELEASE RATE COMPUTATIONS IN A STEADY ROLLING TIRE

Sanjay Govindjee
Structural Engineering, Mechanics, and Materials
Department of Civil Engineering
University of California, Berkeley
Berkeley, CA 94720-1710, USA

## ABSTRACT

The recent well publicized failures of the Firestone ATX/AT tires has brought to light the poor state of research into the basic physical causes of such failures and the poor state of research into analytic methods suitable for analyzing such failures. The essence of problems of this nature lies in the finite deformation fatigue fracture of elastomers at large strains. The definition of failure criteria is further complicated by the issue of material aging. The computational problem is rather demanding due to the truly 3-D nature of the tire system under load. Finding the failure forces (the singular energy momentum tractions) is made challenging by the spinning reference state normally utilized in computational tire analysis. A true analysis of such problems demands that one address all of these issues; this paper focuses on one aspect of this problem – viz., the formulation of suitable analytic expressions for the computation of the crack tip driving forces in spinning tires.

## KEYWORDS

Tire mechanics, Energy release rates, Finite deformation

## INTRODUCTION

In assessing the durability of tires a common physical phenomena to be investigated for steel belted radial passenger tires is belt edge cracking. The important physical quantity to be measured or computed here is the energy release rate. As is well known, energy release rate is the physical quantity introduced by Eshelby to characterize the driving force upon a singularity in the elastic field [2, 3] and it is intimately related to the pioneering fracture studies by Ingles [7], Griffith [6], Irwin [8], and Rivlin and Thomas [15]. In the context of fracture, this driving force or energy release rate is often associated with the J-integral criteria [14, 1]; under common conditions the two are of course synonymous. The application of these basic notions turns out to be a modestly complex undertaking for a spinning tire. If we leave aside the experimental aspects, we have available to us a variety of methods for the computation of energy release rates given mechanical loads. For the tire, one is for practical reasons restricted to methods associated with numerical approximations such as the finite element method[18]; see e.g. [11, 9] for a discussion on computational methods for energy release rates. As noted by Govindjee [4] these methodologies have existed for many years, however, their direct application to the spinning tire in the open peer reviewed literature is amazingly scarce. In this paper we examine the extension of Steinmann's method [16, 17] for finite deformation elasticity to the case of an elastic spinning tire in steady rotation. Analysis in a steady spinning frame of reference is based upon the early work of Lynch [12] and the basic equations for such an analysis in finite deformation were first presented by Oden and Lin [13]; more recently see the discussions of LeTallec and Rahier [10] and Govindjee and Mihalic [5].

## ENERGY RELEASE RATE

The issue of an energy release rate computation comes down to the computation of the following expression:

$$G_A = \lim_{\epsilon \to 0} \int_{\Gamma(\epsilon)} \Sigma^o_{AB} N^o_B \, d\Gamma(\epsilon) \,, \tag{1}$$

where $\Sigma^o_{AB} = W(\boldsymbol{F}^o)\delta_{AB} - F^o_{iA} P^o_{iB}$ is Eshelby's energy-momentum tensor, $W$ is the strain energy density, $F^o_{iA}$ is the deformation gradient, and $P^o_{iA}$ is the first Piola-Kirchhoff stress tensor. The surface $\Gamma(\epsilon)$ with normal $N^o_A$ surrounds the crack tip and is of radius $\epsilon$. The result $G_A$ is the crack tip driving force or energy release rate. Note that the result is zero unless the integrand is singular. Steinmann's method is based on the subtle observation that this quantity is in fact closely related to the "external work" terms in the weak form of the balance equation for the energy-momentum field. In the above and throughout we employ indicial notation with lower case Latin indices for the spatial frame and upper case Latin indices for the reference frame. The superscript $o$ indicates quantities that are associated with the usual continuum mechanics definitions which are expressed in terms of a spatial and referential frame. Later on we will employ greek indices for quantities expressed in a rotating frame and also introduce transformed counterparts to our usual continuum mechanics machinery; these will appear with the same symbols modulo the superscript $o$.

## STRONG FORM OF ENERGY-MOMENTUM BALANCE

The starting point for exploiting the observation of [16, 17] is to look at the governing balance equation for the energy-momentum tensor which can be expressed as:

$$\Sigma^o_{BA,A} - F^o_{iB} B^o_i = -F^o_{iB} \bar{\rho} \ddot{\phi}^o_i \,. \tag{2}$$

In the above, $B^o_i$ represents any body forces (per unit reference volume), $\bar{\rho}$ the reference density of the body, and $\phi^o_i$ is the deformation (motion) map. The relation follows directly from linear momentum balance and an assumption on the smoothness of the motion. In what follows we will assume that $B^o_i = 0$ since the only direct body forces in the tire are gravitational and these can be safely neglected.

## WEAK FORM OF ENERGY-MOMENTUM BALANCE

The weak form of this relation can be expressed with the aid of an arbitrary test function $v_B$ and integration over the reference configuration of the body denoted as $\mathcal{B}$. After integrating by parts one has that

$$\int_{\mathcal{B}} v_{B,A} \Sigma^o_{BA} - \int_{\mathcal{B}} v_B F^o_{iB} \bar{\rho} \ddot{\phi}^o_i = \int_{\partial\mathcal{B}} v_B \Sigma^o_{BA} N^o_A \,. \tag{3}$$

The term on the right-hand side (RHS) is clearly related to the RHS of Eq. (1). In fact, Steinmann makes the observation that the RHS of Eq. (3) is the regular part of the "external virtual work" associated with the balance of energy momentum. For cases where this quantity is zero, the computation of the left-hand side of Eq. (3) will simply give the singular part. The reason the singular term does not appear on the RHS explicitly is that we have assumed no singularities during the application of the divergence theorem. In the case of the tire, the belt edge cracks are open when the cracks are rather short and possibly for points outside of the footprint; in theses cases the RHS will be zero. For longer cracks and in the footprint the crack faces are closed and driven in a shear type mode. In this case, the RHS of Eq. (3) (the non-singular part) will be non-zero. In particular, if we assume frictionless crack faces and a crack face contact pressure $p$, then we will have

$$RHS = \int_{\partial\mathcal{B}} v_B p N^0_A \delta_{AB} + v_B W N^0_B \,. \tag{4}$$

In order to compute the energy release rate (the singular part of the energy-momentum traction) this term needs to be subtracted from the LHS of Eq. (3). The feature that make this methodology attractive is that the machinery necessary for its computation is already built into almost all finite element programs. After solving for the deformed configuration of a body, one merely needs to evaluate the integrals in Eq. (3) to determine the energy release rates for any given crack tip.

## WEAK FORM OF ENERGY-MOMENTUM BALANCE IN A SPINNING FRAME

The main complication in utilizing Eq. (3) is the inertial terms. However, in the case of steady state spinning this obstacle can be over come in a straightforward manner utilizing the formalism developed by Oden and Lin [13]; see also LeTallec and Rahier [10] and Govindjee and Mihalic [5]. In this formalism, a third coordinate system is introduced that rotates with the tire. In this frame of reference points are located by their coordinates $X_\alpha$ where $X_\alpha = R_{\alpha B}(t) X^o_B$ and $R_{\alpha B}(t)$ represents the steady rotation of the tire about its axis of circular symmetry. The mapping from $X_\alpha$ to $x_i$ is given by $\phi_i$. The deformation gradient from this spinning frame of reference to the spatial frame is given by $F_{i\alpha} = F^o_{iA} R_{\alpha A}$. Since we assume the tire is in a steady state rotation we have for the acceleration:

$$\ddot{\phi}^o_i = x_{i,\alpha\beta} \Omega_{\beta\gamma} X_\gamma \Omega_{\alpha\delta} X_\delta + x_{i,\alpha} \Omega_{\alpha\gamma} \Omega_{\gamma\delta} X_\delta \,, \tag{5}$$

where $\Omega_{\alpha\beta} = \dot{R}_{\alpha B} R_{\beta B}$ is the spin rate of the tire. It is also noted that the surface normals are mapped as $N_\alpha = R_{\alpha A} N^o_A$.

For isotropic materials we also have that $W(\boldsymbol{F}^o) = W(\boldsymbol{FR}) = W(\boldsymbol{F})$ and through an abuse of notation that $W(\boldsymbol{C}^o) = W(\boldsymbol{C})$, where $C_{\alpha\beta} = F_{i\alpha} F_{i\beta}$. With these results we can now define an energy-momentum tensor relative to the spinning frame as $\Sigma_{\alpha\beta} = R_{\alpha A} \Sigma^o_{AB} R_{\beta B} = W(\boldsymbol{C})\delta_{\alpha\beta} - C_{\alpha\gamma} S_{\gamma\beta}$, where $S_{\alpha\beta}$ represents the second Piola-Kirchhoff stress tensor measured by the deformation from the rotating frame to the current frame. If now re-examine Eq. (3), we find that the singular part of the energy-momentum traction is given by

$$\begin{aligned} G \;=\; & \int_{\mathcal{B}} R_{\beta B} v_{B,\alpha} \Sigma_{\beta\alpha} - \int_{\partial\mathcal{B}} R_{\beta B} v_B \Sigma_{\beta\alpha} N_\alpha - \int_{\mathcal{B}} \bar{\rho} R_{\beta B} v_B F_{i\beta} \delta_{i\alpha} (\Omega^2)_{\alpha\theta} X_\theta & (6) \\ + & \int_{\mathcal{B}} \bar{\rho} R_{\beta B} v_{B,\delta} F_{i\beta} \Omega_{\alpha\gamma} X_\gamma u_{i,\alpha} \Omega_{\delta\theta} X_\theta + \int_{\mathcal{B}} \bar{\rho} R_{\beta B} v_B F_{i\beta,\delta} \Omega_{\alpha\gamma} X_\gamma u_{i,\alpha} \Omega_{\delta\theta} X_\theta \,, & (7) \end{aligned}$$

where we have assumed that the boundary $\partial\mathcal{B}$ possesses circular symmetry.

To practically execute these integrals we now introduce a finite element approximation by expressing the motion $\phi_i$ as an element of a finite dimension functions space; *ie.* as $\phi_i = \sum_a x^a_i M^a(\boldsymbol{X})$ where $x^a_i$

are the nodal degrees of freedom and $M^a$ is the shape function associated with node $a$. Likewise we approximate the space of test functions such that $v_B = \sum_a v_B^a M^a(\boldsymbol{X})$. From the arbitrariness of the test function we can now express the energy release rate for the steady rotation case upon any node $a$ as:

$$G^a_\beta = \int_{\mathcal{B}} M^a_{,\alpha}\Sigma_{\beta\alpha} - \int_{\partial\mathcal{B}} M^a \Sigma_{\beta\alpha} N_\alpha - \int_{\mathcal{B}} \bar{\rho} M^a F_{i\beta}\delta_{i\alpha}(\Omega^2)_{\alpha\theta}X_\theta \tag{8}$$

$$+ \int_{\mathcal{B}} \bar{\rho} M^a_{,\delta} F_{i\beta}\Omega_{\alpha\gamma}X_\gamma u_{i,\alpha}\Omega_{\delta\theta}X_\theta + \int_{\mathcal{B}} \bar{\rho} M^a F_{i\beta,\delta}\Omega_{\alpha\gamma}X_\gamma u_{i,\alpha}\Omega_{\delta\theta}X_\theta\,, \tag{9}$$

In using these relations one needs to also consider the fact that the tire is composed of materials that are not isotropic. In particular the bead is effectively modeled as transversely isotropic and the steel belts as orthotropic. For the general conditions of transverse isotropy and orthotropy the above results do not hold. However, the anisotropy of the tire is very special in that the tire possess a global geometric invariance with respect to the spin of the tire. In this special case, and in this special case only, the expressions above also hold for general anisotropy without modification.

## APPLICATION

The sequence of computational steps in computing the energy release rate at the tip of a circumferential belt edge crack is as follows. First one models the tire in circular symmetry. This implies that the cross tread grooves can not be modeled; only the circumferential tread grooves can be modeled. For a given tire inflation, speed, and axle force the deformation of the tire is computed by solving the linear momentum balance equations is a steady rotating frame using the method of Oden and Lin [13] as presented by Govindjee and Mihalic [5]. Once the deformation state has been computed it is a straightforward post-processing step to compute $G^a_\beta$ using the relations from the previous section.

As an example consider the tire geometry of the Firestone P235/75R15 AT tire as described in Govindjee [4]. We demonstrate some of the types of analysis that can be performed with the method. Shown in Fig. 1 is the energy release rate at the tip of a 25.5mm circumferential crack for 4 different inflation pressures as a function of angular position where $-\pi/2$ represents the center of the footprint and $\pi/2$ the top of the tire. As can be seen from the figure there is very little effect of inflation pressure over this range of pressures (at 120 kph and a 4.4 kN axle load). Above 0 rad mesh coarsening reduces the accuracy of the reported results. As a second example, shown in Fig. 2 is the effect of belt edge crack length and axle load on the energy release rate increment $G_\beta(-\pi/2) - G_\beta(0)$. The figure also shows some results for the P235/75R15 ATX tire. Clearly such modeling capabilities are essential in analyzing belt edge crack durability issues.

# References


[1] B. Budiansky and Rice. J.R., *Conservation laws and energy-release rates*, Journal of Applied Mechanics **40** (1973), 201–203.

[2] J.D. Eshelby, *The continuum theory of lattice defects*, Solid State Physics, Advances in Research and Applications (F. Seitz and D. Turnbull, eds.), vol. III, Academic Press, 1956, pp. 79–119.

[3] ———, *The elastic energy-momentum tensor*, Journal of Elasticity **5** (1975), 321–335.

[4] S. Govindjee, *Firestone tire failure analysis*, Tech. report, 2001.

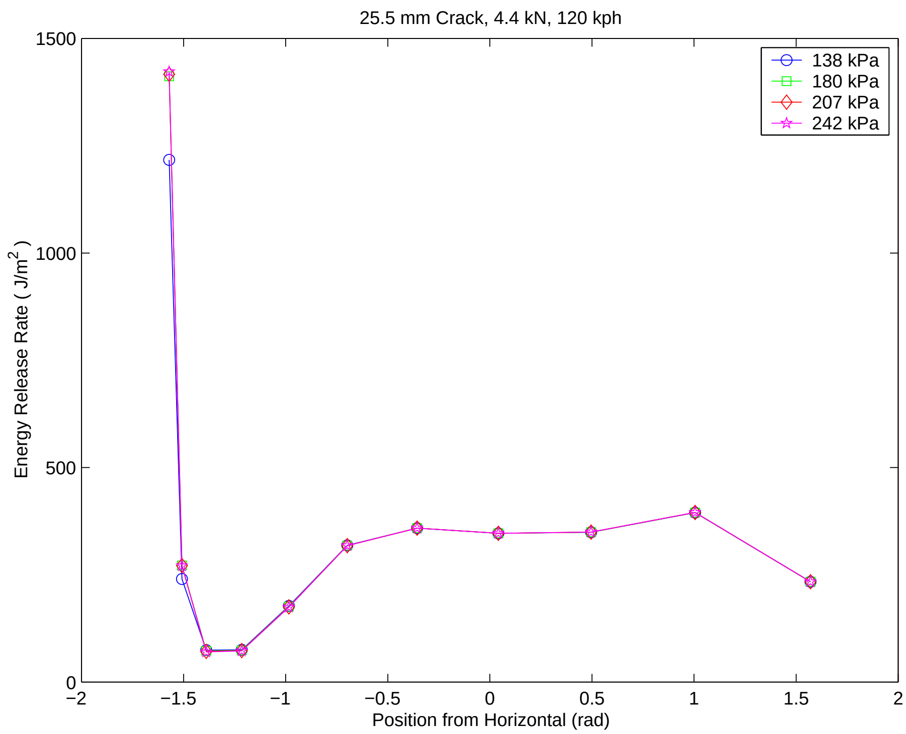


Figure 1: Energy release rate as a function of angular position from the horizontal for a 25.5 mm crack at 120 kph under a 4.4 kN load.

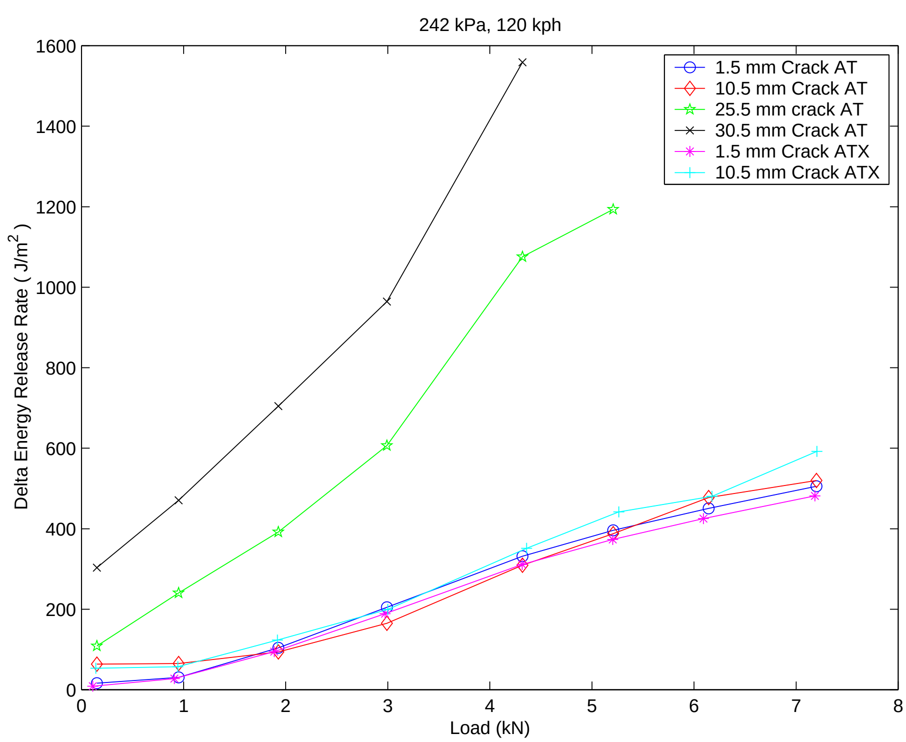


Figure 2: Energy release rate increments per revolution as a function of load indexed by crack length at an inflation pressure of 242 kPa and 120 kph.

[5] S. Govindjee and P.A. Mihalic, *Viscoelastic constitutive relations for the steady spinning of a cylinder*, Tech. Report UCB/SEMM-98/02, University of California Berkeley, Department of Civil Engineering, 1998.

[6] A.A. Griffith, *The phenomena of rupture and flow in solids*, Philosophical Transactions of the Royal Society of London **221A** (1920), 163–197.

[7] C.E. Ingles, *Stresses in a plate due to the presence of cracks and sharp corners*, Transactions of the Institute of Naval Architects **55** (1913), 219–241.

[8] G.R. Irwin, *Fracture dynamics*, pp. 147–166, American Society for Metals, 1948.

[9] K. Kishimoto, S. Aoki, and M. Sakata, *Dynamic stress intensity factors using $\hat{j}$-integral and finite element method*, Engineering Fracture Mechanics **13** (1980), 387–394.

[10] P. Le Tallec and C. Rahier, *Numerical models of steady rolling for non-linear viscoelastic structures in finite deformations*, International Journal for Numerical Methods in Engineering **37** (1994), 1159–1186.

[11] F.Z. Li, C.F. Shih, and A. Needleman, *A comparison of methods for calculating energy release rates*, Engineering Fracture Mechanics **21** (1985), 405–421.

[12] F. DE S. Lynch, *A finite element method of viscoelastic stress analysis with application to rolling contract problems*, International Journal for Numerical Methods in Engineering **1** (1969), 379–394.

[13] J.T. Oden and T.L. Lin, *On the general rolling contact problem for finite deformations of a viscoelastic cylinder*, Computer Methods in Applied Mechanics and Engineering **57** (1986), 297–367.

[14] J.R. Rice, *A path independent integral and the approximate analysis of strain concentration by notches and cracks*, Journal of Applied Mechanics **35** (1968), 379–386.

[15] R.S. Rivlin and A.G. Thomas, *Rupture of rubber 1. Characteristic energy for tearing*, Journal of Polymer Science **10** (1953), 291–318.

[16] P. Steinmann, *Application of material forces to hyperelastic fracture mechanics. I. Continuum mechanical setting*, International Journal of Solids and Structures **37** (2000), 7371–7391.

[17] P. Steinmann, D. Ackermann, and F.J. Barth, *Application of material forces to hyperelastic fracture mechanics. II. Computational setting*, International Journal of Solids and Structures (2001), (to appear).

[18] O.C. Zienkiewicz and R.L. Taylor, *The finite element method, volume 1*, 5 ed., Butterworth and Heinemann, 2000.