ENERGY BALANCE METHOD FOR PREDICTING CRACKING IN CROSS-PLY LAMINATES DURING BEND DEFORMATION

L. N. McCartney and M. J. W. Byrne

NPL Materials Centre, National Physical Laboratory
Queens Road, Teddington, Middlesex, UK, TW11 0LW

INTRODUCTION

Microstructural damage in composite structures arising from the application of mechanical and/or thermal loads is often unavoidable and its effects need to be taken into account when assessing structural performance, especially the occurrence of ply crack formation and delamination. For structures subject to fatigue loading it is desirable to avoid damage occurrence of any kind as it can act as nucleation sites for the growth of macroscopic damage (e.g. delaminations) that eventually leads to the catastrophic failure of the structure. As many structures experience some form of bend deformation during service it is vital that damage formation in the presence of bending is well understood. Of particular relevance to the performance of structures is the prediction of the occurrence of microstructural damage in complex loading modes where out-of-plane bending modes of deformation occur in conjunction with in-plane biaxial and through-thickness loading.

While a great deal of research has been devoted to the case of ply cracking in cross-ply laminates subject only to in-plane deformations, the practically important case of out-of-plane bending has received much less attention (e.g. [1-6]). One objective of this paper is to summarise the important results that have been derived using an energy balance method for predicting the conditions for the steady state growth of ply cracks in a cross-ply laminate subject to bending and thermal residual stresses. A second objective is to indicate how the methodology for ply crack formation can be developed without a detailed analysis of the stress transfer that is in fact needed only to estimate the thermoelastic constants of a damaged laminate (as described in [6]). The anti-clastic (i.e. biaxial) bending typical of deformed laminates will be taken into consideration.

GEOMETRY AND LOADING CONDITIONS

A cross-ply laminate of length 2L, width 2W and total thickness h consisting of perfectly bonded anisotropic layers is considered within a Cartesian coordinate system. The x-direction is taken as the through-thickness direction of the laminate, the y-direction is taken as the axial (longitudinal) direction and the z-direction as the in-plane transverse direction. The bending moments per unit area of loading cross-section for the axial and transverse directions are defined respectively by

$$M = \frac{1}{2hW}\int_{-W}^{W}\int_{0}^{h}\left(x-\frac{h}{2}\right)\sigma_{yy}(x,L,z)\,dx\,dz, \qquad M_T = \frac{1}{2hL}\int_{-L}^{L}\int_{0}^{h}\left(x-\frac{h}{2}\right)\sigma_{zz}(x,y,W)\,dx\,dy. \tag{1}$$

The moments are taken about the mid-plane of the laminate which might not correspond to the neutral axis if the laminate is unsymmetrical and/or damaged in the region of axial tension. The corresponding effective applied axial and transverse stresses are defined respectively by

$$\sigma = \frac{1}{2hW}\int_{-W}^{W}\int_{0}^{h}\sigma_{yy}(x,L,z)\,dxdz\,, \qquad \sigma_T = \frac{1}{2hL}\int_{-L}^{L}\int_{0}^{h}\sigma_{zz}(x,y,W)\,dxdy\,. \tag{2}$$

The faces of the laminate are assumed to be subject to a uniform applied tensile traction σ_t. The corresponding effective through-thickness strain ε_t for a laminate is defined by

$$\varepsilon_t = \frac{1}{4hLW}\int_{-W}^{W}\int_{-L}^{L}\left[u(h,y,z) - u(0,y,z)\right]dydz\,, \tag{3}$$

and the effective applied in-plane axial and transverse strains $\bar{\varepsilon}$ and $\bar{\varepsilon}_T$ are defined by

$$\begin{aligned} \bar{\varepsilon} &= \frac{1}{4hLW}\int_{-W}^{W}\int_{0}^{h}\left[v(x,L,z) - v(x,-L,z)\right]dx\,dz, \\ \bar{\varepsilon}_T &= \frac{1}{4hLW}\int_{-L}^{L}\int_{0}^{h}\left[w(x,y,W) - w(x,y,-W)\right]dx\,dy, \end{aligned} \tag{4}$$

where u, v and w are respectively the x, y and z-components of the displacement vector.

In order to represent bending of the laminate for the axial and transverse directions it is assumed that the edge boundary conditions for the displacement components v and w are of the form

$$v = \begin{cases} L(\varepsilon + \hat{\varepsilon}x) & \text{on } x = L \\ -L(\varepsilon + \hat{\varepsilon}x) & \text{on } x = -L \end{cases} \qquad w = \begin{cases} W(\varepsilon_T + \hat{\varepsilon}_T x) & \text{on } z = W \\ -W(\varepsilon_T + \hat{\varepsilon}_T x) & \text{on } z = -W\,. \end{cases} \tag{5}$$

For infinitesimal deformations the radii of curvatures of the surface x = 0 of the deformed laminate are given by $R_1 = 1/|\hat{\varepsilon}|$ and $R_2 = 1/|\hat{\varepsilon}_T|$, so that $\hat{\varepsilon}$ and $\hat{\varepsilon}_T$ are respectively the curvatures of the surface x = 0 of the deformed laminate in the axial and transverse directions. Substituting the edge boundary conditions (5) into (4) and performing the integrations leads to

$$\bar{\varepsilon} = \varepsilon + \tfrac{1}{2}h\,\hat{\varepsilon}\,, \qquad \bar{\varepsilon}_T = \varepsilon_T + \tfrac{1}{2}h\,\hat{\varepsilon}_T\,. \tag{6}$$

It is clear that $\bar{\varepsilon}$ and $\bar{\varepsilon}_T$ are the axial and transverse strains on the mid-plane of the laminate subject to the edge conditions (5).

STRESS-STRAIN RELATIONS

It has been shown [6] that the effective stress-strain relations for a damaged cross-ply laminate are of the following form

$$\varepsilon_t = \frac{\sigma_t}{E_t} - \frac{\nu_a}{E_A}\sigma - \frac{\nu_t}{E_T}\sigma_T - \frac{\hat{\nu}_a}{\hat{E}_A}M - \frac{\hat{\eta}_t}{\hat{E}_T}M_T + \alpha_t\Delta T\,, \tag{7}$$

$$\bar{\varepsilon} = -\frac{\nu_a}{E_A}\sigma_t + \frac{\sigma}{E_A} - \frac{\nu_A}{E_A}\sigma_T - \frac{\hat{\nu}_A}{\hat{E}_A}M - \frac{\hat{\eta}_A}{\hat{E}_T}M_T + \alpha_A\Delta T\,, \tag{8}$$

$$\bar{\varepsilon}_T = -\frac{\nu_t}{E_T}\sigma_t - \frac{\nu_A}{E_A}\sigma + \frac{\sigma_T}{E_T} - \frac{\hat{\nu}_T}{\hat{E}_A}M - \frac{\hat{\eta}_T}{\hat{E}_T}M_T + \alpha_T\Delta T\,, \tag{9}$$

$$\hat{\varepsilon} = -\frac{\hat{\nu}_a}{\hat{E}_A}\sigma_t - \frac{\hat{\nu}_A}{\hat{E}_A}\sigma - \frac{\hat{\nu}_T}{\hat{E}_A}\sigma_T + \frac{M}{\hat{E}_A} - \frac{\hat{\delta}_A}{\hat{E}_A}M_T + \hat{\alpha}_A\Delta T\,, \tag{10}$$

$$\hat{\varepsilon}_T = -\frac{\hat{\eta}_t}{\hat{E}_T}\sigma_t - \frac{\hat{\eta}_A}{\hat{E}_T}\sigma - \frac{\hat{\eta}_T}{\hat{E}_T}\sigma_T - \frac{\hat{\delta}_A}{\hat{E}_A}M + \frac{M_T}{\hat{E}_T} + \hat{\alpha}_T\Delta T\,, \tag{11}$$

which defines the various thermo-elastic constants that characterise the properties of a damaged laminate subject to combined in-plane biaxial loading, out-of-plane through-thickness loading and bending.

REDUCED STRESS-STRAIN RELATIONS FOR CONSTRAINED TRIAXIAL LOADING

The damage-dependent stress-strain relations (10) and (11) are treated as a linear system of algebraic equations with unknowns M and M_T. For convenience, the following parameters are defined

$$\hat{\delta}_T = \hat{\delta}_A \frac{\hat{E}_T}{\hat{E}_A}, \qquad \Lambda = 1 - \hat{\delta}_A \hat{\delta}_T \ . \tag{12}$$

On solving (10) and (11) for M and M_T and substituting into the remaining damage-dependent stress-strain relations (7-9), the following reduced stress-strain relations are derived of the same form as those for a cross-ply laminate subject only to triaxial loading, without shear or bending

$$\tilde{\varepsilon}_t = \varepsilon_t + \frac{\hat{\nu}_a}{\Lambda}\left[\hat{\varepsilon} + \hat{\delta}_T \hat{\varepsilon}_T\right] + \frac{\hat{\eta}_t}{\Lambda}\left[\hat{\varepsilon}_T + \hat{\delta}_A \hat{\varepsilon}\right] = \frac{\sigma_t}{\tilde{E}_t} - \frac{\tilde{\nu}_a}{\tilde{E}_A}\sigma - \frac{\tilde{\nu}_t}{\tilde{E}_T}\sigma_T + \tilde{\alpha}_t \Delta T \ , \tag{13}$$

$$\tilde{\varepsilon} = \bar{\varepsilon} + \frac{\hat{\nu}_A}{\Lambda}\left[\hat{\varepsilon} + \hat{\delta}_T \hat{\varepsilon}_T\right] + \frac{\hat{\eta}_A}{\Lambda}\left[\hat{\varepsilon}_T + \hat{\delta}_A \hat{\varepsilon}\right] = -\frac{\tilde{\nu}_a}{\tilde{E}_A}\sigma_t + \frac{\sigma}{\tilde{E}_A} - \frac{\tilde{\nu}_A}{\tilde{E}_A}\sigma_T + \tilde{\alpha}_A \Delta T \ , \tag{14}$$

$$\tilde{\varepsilon}_T = \bar{\varepsilon}_T + \frac{\hat{\nu}_T}{\Lambda}\left[\hat{\varepsilon} + \hat{\delta}_T \hat{\varepsilon}_T\right] + \frac{\hat{\eta}_T}{\Lambda}\left[\hat{\varepsilon}_T + \hat{\delta}_A \hat{\varepsilon}\right] = -\frac{\tilde{\nu}_t}{\tilde{E}_T}\sigma_t - \frac{\tilde{\nu}_A}{\tilde{E}_A}\sigma + \frac{\sigma_T}{\tilde{E}_T} + \tilde{\alpha}_T \Delta T, \tag{15}$$

where the reduced strains $\tilde{\varepsilon}_t$, $\tilde{\varepsilon}$ and $\tilde{\varepsilon}_T$ can be interpreted as strains for a damaged laminate, subject to triaxial loading and constrained so that bending strains are zero, and where the reduced thermoelastic constants are defined by

$$\frac{1}{\tilde{E}_t} = \frac{1}{E_t} - \frac{1}{\Lambda}\left\{\frac{\hat{\nu}_a^2}{\hat{E}_A} + \frac{\hat{\eta}_t^2}{\hat{E}_T} + \frac{2\hat{\delta}_A \hat{\nu}_a \hat{\eta}_t}{\hat{E}_A}\right\}, \qquad \frac{1}{\tilde{E}_A} = \frac{1}{E_A} - \frac{1}{\Lambda}\left\{\frac{\hat{\nu}_A^2}{\hat{E}_A} + \frac{\hat{\eta}_A^2}{\hat{E}_T} + \frac{2\hat{\delta}_A \hat{\nu}_A \hat{\eta}_A}{\hat{E}_A}\right\},$$

$$\frac{1}{\tilde{E}_T} = \frac{1}{E_T} - \frac{1}{\Lambda}\left\{\frac{\hat{\nu}_T^2}{\hat{E}_A} + \frac{\hat{\eta}_T^2}{\hat{E}_T} + \frac{2\hat{\delta}_A \hat{\nu}_T \hat{\eta}_T}{\hat{E}_A}\right\}, \qquad \frac{\tilde{\nu}_a}{\tilde{E}_A} = \frac{\nu_a}{E_A} + \frac{1}{\Lambda}\left\{\frac{\hat{\nu}_a}{\hat{E}_A}\left(\hat{\nu}_A + \hat{\delta}_A \hat{\eta}_A\right) + \frac{\hat{\eta}_t}{\hat{E}_T}\left(\hat{\delta}_T \hat{\nu}_A + \hat{\eta}_A\right)\right\},$$

$$\frac{\tilde{\nu}_t}{\tilde{E}_T} = \frac{\nu_t}{E_T} + \frac{1}{\Lambda}\left\{\frac{\hat{\nu}_a}{\hat{E}_A}\left(\hat{\nu}_T + \hat{\delta}_A \hat{\eta}_T\right) + \frac{\hat{\eta}_t}{\hat{E}_T}\left(\hat{\delta}_T \hat{\nu}_T + \hat{\eta}_T\right)\right\},$$

$$\frac{\tilde{\nu}_A}{\tilde{E}_A} = \frac{\nu_A}{E_A} + \frac{1}{\Lambda}\left\{\frac{\hat{\nu}_A}{\hat{E}_A}\left(\hat{\nu}_T + \hat{\delta}_A \hat{\eta}_T\right) + \frac{\hat{\eta}_A}{\hat{E}_T}\left(\hat{\delta}_T \hat{\nu}_T + \hat{\eta}_T\right)\right\},$$

$$\tilde{\alpha}_t = \alpha_t + \frac{1}{\Lambda}\left[\hat{\nu}_a\left(\hat{\delta}_T \hat{\alpha}_T + \hat{\alpha}_A\right) + \hat{\eta}_t\left(\hat{\alpha}_T + \hat{\delta}_A \hat{\alpha}_A\right)\right], \quad \tilde{\alpha}_A = \alpha_A + \frac{1}{\Lambda}\left[\hat{\nu}_A\left(\hat{\delta}_T \hat{\alpha}_T + \hat{\alpha}_A\right) + \hat{\eta}_A\left(\hat{\alpha}_T + \hat{\delta}_A \hat{\alpha}_A\right)\right],$$

$$\tilde{\alpha}_T = \alpha_T + \frac{1}{\Lambda}\left[\hat{\nu}_T\left(\hat{\delta}_T \hat{\alpha}_T + \hat{\alpha}_A\right) + \hat{\eta}_T\left(\hat{\alpha}_T + \hat{\delta}_A \hat{\alpha}_A\right)\right].$$

The corresponding reduced stress-strain relations for an undamaged laminate are written:

$$\tilde{\varepsilon}_t^o = \frac{\sigma_t}{\tilde{E}_t^o} - \frac{\tilde{\nu}_a^o}{\tilde{E}_A^o}\sigma - \frac{\tilde{\nu}_t^o}{\tilde{E}_T^o}\sigma_T + \tilde{\alpha}_t^o \Delta T, \tag{16}$$

$$\tilde{\varepsilon}^o = -\frac{\tilde{\nu}_a^o}{\tilde{E}_A^o}\sigma_t + \frac{\sigma}{\tilde{E}_A^o} - \frac{\tilde{\nu}_A^o}{\tilde{E}_A^o}\sigma_T + \tilde{\alpha}_A^o \Delta T \ , \tag{17}$$

$$\tilde{\varepsilon}_T^o = -\frac{\tilde{\nu}_t^o}{\tilde{E}_T^o}\sigma_t - \frac{\tilde{\nu}_A^o}{\tilde{E}_A^o}\sigma + \frac{\sigma_T}{\tilde{E}_T^o} + \tilde{\alpha}_T^o \Delta T, \tag{18}$$

where a superscript 'o' denotes that the strains and laminate properties refer to their values for the undamaged state of the laminate.

FUNDAMENTAL INTER-RELATIONSHIPS BETWEEN THERMO-ELASTIC CONSTANTS

By considering the conditions for ply crack closure during uniaxial loading in the axial, transverse and through-thickness directions, it can be shown [7] that many inter-relationships between the thermoelastic constants of a damaged laminate can be derived. First of all define

$$\Phi = \frac{1}{\tilde{E}_A} - \frac{1}{\tilde{E}_A^o}. \tag{19}$$

It has been shown [7] that the thermo-elastic constants for a damaged laminate are related to those of the corresponding undamaged laminate according to the following simple relations

$$\frac{1}{\tilde{E}_t} - \frac{1}{\tilde{E}_t^o} = (k')^2 \Phi\,, \qquad \frac{1}{\tilde{E}_T} - \frac{1}{\tilde{E}_T^o} = k^2 \Phi\,, \tag{20}$$

$$\frac{\tilde{\nu}_t^o}{\tilde{E}_T^o} - \frac{\tilde{\nu}_t}{\tilde{E}_T} = k\,k'\,\Phi\,, \qquad \frac{\tilde{\nu}_A^o}{\tilde{E}_A^o} - \frac{\tilde{\nu}_A}{\tilde{E}_A} = k\,\Phi\,, \qquad \frac{\tilde{\nu}_a^o}{\tilde{E}_A^o} - \frac{\tilde{\nu}_a}{\tilde{E}_A} = k'\,\Phi\,, \tag{21}$$

$$\tilde{\alpha}_t - \tilde{\alpha}_t^o = k'\,k_1\Phi\,, \qquad \tilde{\alpha}_A - \tilde{\alpha}_A^o = k_1\Phi\,, \qquad \tilde{\alpha}_T - \tilde{\alpha}_T^o = k\,k_1\Phi\,, \tag{22}$$

where the constants k_1, k and k' are easily obtained from the geometry and ply properties of the undamaged laminate. The results (20-22) indicate that the degradation of all the thermoelastic constants of a damaged laminate arising from ply cracking in the 90° plies can be characterised by a single parameter Φ that is defined at the macroscopic laminate level by (19).

GIBBS FREE ENERGY FOR A CRACKED LAMINATE SUBJECT TO MULTI-AXIAL BENDING

It has been shown [7], for the case of uniform ply crack densities in one or more of the 90° plies of the laminate, that the Gibbs free energy (equivalent to the complementary energy) per unit volume of laminate (averaged over the region V occupied by the laminate) may be expressed in the form

$$\begin{aligned}
\bar{g} = &-\frac{\sigma_t^2}{2E_t} - \frac{\sigma^2}{2E_A} - \frac{\sigma_T^2}{2E_T} - \frac{M^2}{2\hat{E}_A} - \frac{M_T^2}{2\hat{E}_T} + \frac{\nu_a}{E_A}\sigma_t\sigma + \frac{\nu_t}{E_T}\sigma_t\sigma_T + \frac{\nu_A}{E_A}\sigma\sigma_T \\
&+ \frac{\hat{\nu}_a}{\hat{E}_A}\sigma_t M + \frac{\hat{\nu}_A}{\hat{E}_A}\sigma M + \frac{\hat{\nu}_T}{\hat{E}_A}\sigma_T M + \frac{\hat{\eta}_t}{\hat{E}_T}\sigma_t M_T + \frac{\hat{\eta}_A}{\hat{E}_T}\sigma M_T + \frac{\hat{\eta}_T}{\hat{E}_T}\sigma_T M_T + \frac{\hat{\delta}_A}{\hat{E}_A} M M_T \\
&- \left[\sigma_t\alpha_t + \sigma\alpha_A + \sigma_T\alpha_T + M\hat{\alpha}_A + M_T\hat{\alpha}_T\right]\Delta T \\
&+ \frac{1}{2}\left[\sigma_t^c\alpha_t + \sigma^c\alpha_A + \sigma_T^c\alpha_T + M^c\hat{\alpha}_A + M_T^c\hat{\alpha}_T\right]\Delta T - \frac{1}{2V}\int_V \sigma_{ij}^c\,\alpha_{ij}\Delta T\,dV + g_o(\Delta T).
\end{aligned} \tag{23}$$

It should be noted that the stress-strain relations (7-11) for a damaged laminate may be obtained from (23) by differentiating with respect to $\sigma_t, \sigma, \sigma_T, M$ and M_T. It has been shown [7] that, on using (19-22) the complicated result (23) can be reduced to the simple form

$$\bar{g} - \bar{g}_0 = -\frac{\Phi}{2}\left[k'\sigma_t + \sigma + k\sigma_T - \bar{\sigma}^c\right]^2 - F(\hat{\varepsilon}, \hat{\varepsilon}_T) + F_0(\hat{\varepsilon}^o, \hat{\varepsilon}_T^o)\,, \tag{24}$$

where $\bar{g}_0$ is the value of $\bar{g}$ for an undamaged laminate, $\bar{\sigma}^c$ is the crack closure stress for uniaxial in-plane loading constrained so that there is no bending, and where

$$
\begin{aligned}
F(\hat{\varepsilon},\,\hat{\varepsilon}_T) &= \frac{1}{2\Lambda}\Big[\,\hat{E}_A\hat{\varepsilon}\left(\hat{\varepsilon}+\hat{\delta}_T\hat{\varepsilon}_T\right)+\hat{E}_T\hat{\varepsilon}_T\left(\hat{\varepsilon}_T+\hat{\delta}_A\hat{\varepsilon}\right)\Big], \qquad \Lambda = 1-\hat{\delta}_A\hat{\delta}_T\,, \\
F_0(\hat{\varepsilon},\,\hat{\varepsilon}_T) &= \frac{1}{2\Lambda_0}\Big[\,\hat{E}^o_A\hat{\varepsilon}\left(\hat{\varepsilon}+\hat{\delta}^o_T\hat{\varepsilon}_T\right)+\hat{E}^o_T\hat{\varepsilon}_T\left(\hat{\varepsilon}_T+\hat{\delta}^o_A\hat{\varepsilon}\right)\Big], \qquad \Lambda_0 = 1-\hat{\delta}^o_A\hat{\delta}^o_T\,.
\end{aligned}
\tag{25}
$$

PREDICTING DAMAGE FORMATION

Consider the special case where, during the formation of every ply crack, the fracture energy for ply crack formation has a unique value 2γ. The first objective is to determine the conditions for which it is energetically favourable for an array of equally spaced ply cracks having density ρ_0 to form quasi-statically in an undamaged laminate subject to fixed applied loads and temperature. For a macroscopic region V of the laminate, energy balance considerations and the fact that kinetic energy is never negative, lead to the criterion for crack formation having the form [8]

$$\Delta\Gamma \;+\; \Delta G \;<\; 0\;, \tag{26}$$

where the energy absorbed in a macroscopic volume V of laminate by the formation of the new ply cracks is given by

$$\Delta\Gamma \;=\; V\Gamma \;=\; V\,\frac{2\gamma\rho_0 h^{(90)}}{h}\,. \tag{27}$$

In (27) the parameter Γ denotes the energy absorbed per unit volume of laminate during the formation of new ply crack surfaces in the 90^o plies that have led to the initial damage state denoted by the ply crack density ρ_0, and $h^{(90)}$ is the total thickness of the 90^o plies in which the ply cracks have formed. The corresponding change of Gibbs free energy in the region V of the laminate is

$$\Delta G \;=\; \int_V \left[\,g-g_0\right]dV \;=\; V\left[\,\overline{g}-\overline{g}_0\right]. \tag{28}$$

In (28) g denotes the Gibbs free energy per unit volume when the damage in the laminate is characterised by the ply crack density ρ_0, and g_o denotes the corresponding value of g in the undamaged state. It follows from (24) that

$$\overline{g}-\overline{g}_0 = -\,\frac{\Phi(\rho_0)}{2}\left[k'\sigma_t+\sigma+k\sigma_T-\overline{\sigma}^c\right]^2 \;-\; F(\hat{\varepsilon},\,\hat{\varepsilon}_T,\,\rho_0) \;+\; F_0(\hat{\varepsilon}^o,\,\hat{\varepsilon}^o_T)\;, \tag{29}$$

where from (19)

$$\Phi(\rho_0) = \frac{1}{\tilde{E}_A(\rho_0)} - \frac{1}{\tilde{E}^o_A}\;, \tag{30}$$

and from $(25)_1$

$$
\begin{aligned}
F(\hat{\varepsilon},\,\hat{\varepsilon}_T,\,\rho) &= \frac{1}{2\Lambda(\rho)}\Big[\,\hat{E}_A(\rho)\hat{\varepsilon}\left(\hat{\varepsilon}+\hat{\delta}_T(\rho)\hat{\varepsilon}_T\right)+\hat{E}_T(\rho)\hat{\varepsilon}_T\left(\hat{\varepsilon}_T+\hat{\delta}_A(\rho)\hat{\varepsilon}\right)\Big]\,, \\
\Lambda(\rho) &= 1-\hat{\delta}_A(\rho)\hat{\delta}_T(\rho)\;.
\end{aligned}
\tag{31}
$$

It then follows using (26-29) that ply crack formation is governed by the inequality

$$\frac{\Phi(\rho_0)}{2}\left[k'\sigma_t+\sigma+k\sigma_T-\overline{\sigma}^c\right]^2 \;+\; F(\hat{\varepsilon},\,\hat{\varepsilon}_T,\,\rho_0) \;-\; F_0(\hat{\varepsilon}^o,\,\hat{\varepsilon}^o_T) \;>\; \frac{2\gamma\rho_0 h^{(90)}}{h}\;. \tag{32}$$

It is emphasised that the approach described above does not provide any information that indicates how the thermo-elastic constants depend upon ply crack density. A detailed stress analysis is required to provide this information (see [6]). Progressive ply crack formation can be analysed in a very similar way as described in [7].

CONCLUSIONS

The methodology that has been described for the prediction of ply crack formation in some or all of the 90^o plies of a multiple-ply cross-ply laminate subject to combined in-plane biaxial loading, out-of-plane through-thickness loading and bending about two orthogonal axes, and thermal residual stresses, has the following properties:

- the analysis is exact within the assumptions made,
- the stress-strain relations for a damaged laminate may be obtained by differentiating the Gibbs free energy with respect to the loading parameters, and the form of the stress-strain relations is identical to that for undamaged laminates,
- the consideration of ply crack closure conditions leads to the important result that the degradation of the properties of the thermo-elastic constants of the laminate is governed by a single parameter Φ that is dependent only on the axial moduli of the laminate in both the damaged and undamaged states,
- the form of the property degradation relations enables the complex expression for the Gibbs free energy of a damaged laminate to be written in a simple form,
- the results enable the derivation of a relatively simple criterion for the prediction of the loading and thermal conditions that are energetically favourable for ply cracks to form in some or all of the 90^o plies,
- the ply crack formation criterion may be used to develop a methodology for the prediction of progressive ply crack formation during loading.

REFERENCES

1. Li S, Reid S R & Soden P D, 'A finite strip analysis of cracked laminates', *Mech. Mater.* 18, (1994), 289-311.
2. McCartney L N & Pierse C, 'Stress transfer mechanics for multiple-ply laminates subject to bending', NPL Report CMMT(A)55, February 1997.
3. McCartney L N & Pierse C, 'Stress transfer mechanics for multiple ply laminates for axial loading and bending', Proc. 11th Int. Conf. on Composite Materials, Gold Coast, Australia, July 14-18, 1997, vol. V, 662-671.
4. Smith P A and Ogin S L, 'Characterisation and modelling of matrix cracking in a $(0/90)_{2s}$ GFRP laminate loaded in flexure', *Proc. Roy. Soc. Lond.* **A 456**, (2000), 2755 - 2770.
5. Smith P A and Ogin S L, 'On transverse matrix cracking in cross-ply laminates loaded in simple bending', *Composites Part A*, **30**, (1999), 1003 - 1008.
6. McCartney L N, 'Stress transfer mechanics for multiple ply cross-ply laminates subject to bending', summary in Proceedings of 6th International Conference on the Deformation and Fracture of Composites, Manchester, April 2001, pp. 57-66. Full details to be published.
7. McCartney L N & Byrne M J W, 'Energy balance method of predicting ply crack formation during bend deformation', to be published.
8. McCartney L N, 'Prediction of microcracking in composite materials', in "FRACTURE: A Topical Encyclopaedia of Current Knowledge Dedicated to Alan Arnold Griffith", (ed. G.P.Cherepanov), Krieger Publishing Company, Melbourne, USA, 1998, 905-916.