ORAL/POSTER REFERENCE: ICF 100177OR

DIRECT EVALUATION OF ACCURATE SIF WITH PUM

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ABSTRACT

An alternative approach to the extended finite element method (XFEM) and generalised finite element method (GFEM) is introduced to enrich the finite element approximation of the crack tip node as well as its surrounding nodes. These nodes are enriched with not only the first term but also the higher order terms of the crack tip asymptotic field using a partition of unity method (PUM). The first term only is used in the XFEM to enrich the surrounding nodes, and in the GFEM to enrich the crack tip node. This approach also differs from the XFEM in that the additional coefficients of the enriched nodes are the actual coefficients of the crack tip asymptotic field. Numerical results show that together with a reduced quadrature rule, the current approach predicts accurate stress intensity factors directly after constraining the enriched nodes properly but without extra post-processing.

KEYWORDS

Crack tip asymptotic field, partition of unity method (PUM), stress intensity factor (SIF)

INTRODUCTION

For crack problems, Tong and Pian [1] have shown that, in general, the convergence rate for the finite element (FE) method is dominated by the nature of the solution near the point of singularity, and the error from the elements immediately adjacent to the point is of the same order as that of the remainder of the elements. Hence neither the use of the regular high accuracy elements using high order polynomials as interpolation functions nor finer elements improve the accuracy efficiently. In order to improve the convergence rate of the FE solution, various singular elements have been introduced to account for the required crack tip singularity [2-4]. Recently, Belytschko et al. [5-7] proposed the extended finite element method (XFEM) for modelling cracks in the FE framework, which seems promising for fracture problems since it avoids using a mesh conforming with the crack as is the case with the traditional FEM. By using XFEM, a standard FE mesh for the problem is first created without accounting for the crack. A crack is then represented independently of the mesh by enriching the standard displacement approximation with both discontinuous displacement fields along the crack face and the singular asymptotic fields at nodes surrounding the crack tip through a partition of unity method (PUM). The additional coefficients at each enriched node are independent. Strouboulis et al. [8] also discussed the possibility of enriching the crack tip node with the asymptotic field in their generalised finite element method (GFEM). The difference between the XFEM and the GFEM is that the former enriches the surrounding nodes instead of the crack tip, while the latter only enriches the crack tip. The weakness of most singular elements as well as the XFEM and the

GFEM is that they predict accurate global displacements but not accurate SIFs at the crack tip. The SIF has to be evaluated with the help of energy related parameters such as the *J*-integral by a post-processing procedure. This limits the application of singular elements, XFEM or GFEM in fracture simulation.

In order to determine the SIF directly without extra post-processing, an alternative approach is introduced to enrich the FE approximation of the crack tip node as well as its surrounding nodes with not only the first term but also the higher order terms of the crack tip asymptotic field using the PUM. It differs from the XFEM in that the enriched fields are the actual crack tip asymptotic fields and the additional coefficients of the enriched nodes are the relevant coefficients of this expansion. Sensitivity to the quadrature rule and number of retained terms, as well as the effect of constraining the enriched nodes, are studied. The computed SIFs of typical cracked specimens will be validated with results available in the literature.

ENRICHING THE CRACK TIP FE APPROXIMATION USING PUM

For our purposes and without loss of generality, we consider only Mode I crack problems. The truncated N terms of the displacement expansions near the tip of a crack can be written as [3, 4]

$$u_a(r,\theta)=\sum_{n=1}^{N} a_n f_{1n}(r,\theta), \qquad v_a(r,\theta)=\sum_{n=1}^{N} a_n f_{2n}(r,\theta) \tag{1}$$

where (r, θ) are the polar coordinates with the origin at the crack tip, and the angular functions

$$f_{1n}(r,\theta)=\frac{r^{n/2}}{2\mu}\left[\left(\kappa+\frac{n}{2}+(-1)^n\right)\cos\frac{n}{2}\theta-\frac{n}{2}\cos\left(\frac{n}{2}-2\right)\theta\right] \tag{2}$$

$$f_{2n}(r,\theta)=\frac{r^{n/2}}{2\mu}\left[\left(\kappa-\frac{n}{2}-(-1)^n\right)\sin\frac{n}{2}\theta+\frac{n}{2}\sin\left(\frac{n}{2}-2\right)\theta\right] \tag{3}$$

The coefficient of the first term is related to the mode I SIF K_I as $a_1 = K_I/\sqrt{2\pi}$.

In order to use higher order terms, r is normalised as

$$\bar{r} = r/r_m \tag{4}$$

where r_m is a characteristic length of the elements with enriched nodes, e.g., the length of a side or of the diagonal of a rectangular element. Taking into account of (4), displacements (1) become

$$u_a(\bar{r},\theta)=\sum_{n=1}^{N} a'_n f_{1n}(\bar{r},\theta), \qquad v_a(\bar{r},\theta)=\sum_{n=1}^{N} a'_n f_{2n}(\bar{r},\theta) \tag{5}$$

with the coefficients being related as

$$a'_n = a_n r_m^{n/2} \tag{6}$$

For an element near a crack tip (cf. Figure 1), the approximation of displacements enriched with the truncated crack tip asymptotic fields (5) using the PUM can be written as

$$\begin{Bmatrix} u^h(x) \\ v^h(x) \end{Bmatrix} = \sum_{i\in I}\phi_i(x)\begin{Bmatrix} u_i \\ v_i \end{Bmatrix} + \sum_{j\in E\cap I}\phi_j(x)\begin{Bmatrix} u_a(\bar{r},\theta) \\ v_a(\bar{r},\theta) \end{Bmatrix} = \sum_{i\in I}\phi_i(x)\begin{Bmatrix} u_i \\ v_i \end{Bmatrix} + \sum_{j\in E\cap I}\sum_{n\in N}\phi_j(x)\begin{Bmatrix} f_{1n}(\bar{r},\theta) \\ f_{2n}(\bar{r},\theta) \end{Bmatrix} a_n \tag{7}$$

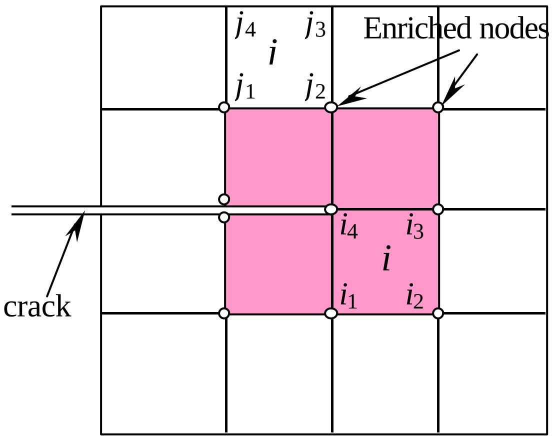

Figure 1: A schematic picture of the elements and enriched nodes near a crack tip

where I is the node set of an element, e.g., for elements i and j in Figure 1, $I=\{i_1 \quad i_2 \quad i_3 \quad i_4\}$ and $\{j_1 \quad j_2 \quad j_3 \quad j_4\}$, respectively. E is the set of the enriched nodes of the element, $E=\{i_1 \quad i_2 \quad i_3 \quad i_4\}$ for element i, and $E=\{j_1 \quad j_2\}$ for element j. Approximation (7) may be simplified in actual cases. For an element which includes the crack tip, e.g. element i in Figure 1, I=E. Noting the consistency condition $\sum_{i\in I} f_i(x)=1$ we have from (7) the displacement approximation

$$\begin{Bmatrix} u^h(x) \\ v^h(x) \end{Bmatrix} = \sum_{i\in I} f_i(x) \begin{Bmatrix} u_i \\ v_i \end{Bmatrix} + \sum_{n\in N} \begin{Bmatrix} f_{1n}(\bar{r},q) \\ f_{2n}(\bar{r},q) \end{Bmatrix} a'_n \tag{8}$$

While for an element which does not include the crack tip, e.g. element j, we have

$$\begin{Bmatrix} u^h(x) \\ v^h(x) \end{Bmatrix} = \sum_{i\in I} f_i(x) \begin{Bmatrix} u_i \\ v_i \end{Bmatrix} + \left(f_{j_1}(x) + f_{j_2}(x)\right) \sum_{n\in N} \begin{Bmatrix} f_{1n}(\bar{r},q) \\ f_{2n}(\bar{r},q) \end{Bmatrix} a'_n \tag{9}$$

If all enriched nodes are constrained, i.e., the nodal displacements are set to be zero, the displacement approximation for elements including the crack tip (e.g., shaded elements in Figure 1) become the truncated crack tip asymptotic field. The outer ring of elements surrounding these elements actually match the crack tip field with the standard FE approximation. However, the current approach differs from all existing singular elements in that higher order terms have been taken into account.

From (7), we have the strain vector

$$\begin{Bmatrix} e_x \\ e_y \\ g_{xy} \end{Bmatrix} = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{Bmatrix} u^h(x) \\ v^h(x) \end{Bmatrix} = Bq + \bar{B}a' \tag{10}$$

where q is the general nodal displacement vector, and $a'^T = \{a'_1 \quad a'_2 \quad \cdots \quad a'_N\}$. The additional matrix

$$\bar{B} = \begin{bmatrix} \bar{B}_1 & \bar{B}_2 & \cdots & \bar{B}_n & \cdots & \bar{B}_N \end{bmatrix} \tag{11}$$

with its nth element or column being

$$\overline{B}_n = \sum_{j \in E \cap I} \begin{bmatrix} \frac{\partial}{\partial x}\left(f_j(x) f_{1n}(\bar{r}, q)\right) \\ \frac{\partial}{\partial y}\left(f_j(x) f_{2n}(\bar{r}, q)\right) \\ \frac{\partial}{\partial y}\left(f_j(x) f_{1n}(\bar{r}, q)\right) + \frac{\partial}{\partial x}\left(f_j(x) f_{2n}(\bar{r}, q)\right) \end{bmatrix} \tag{12}$$

With the use of the enriched strain-displacement relation (10), the element stiffness matrix can be formed in the general way.

NUMERICAL EXAMPLES

A single edge crack in a finite rectangular plate (SEC) shown in Figure 2 is chosen first as a benchmark problem. The coefficient of the singular term a_1 computed by the present method will be compared with the reference K_I solution given in [9]

$$\frac{K_I}{K_0} = 1.12 - 0.23\frac{c}{b} + 10.6\left(\frac{c}{b}\right)^2 - 21.7\left(\frac{c}{b}\right)^3 + 30.4\left(\frac{c}{b}\right)^4, \qquad K_0 = s\sqrt{pc} \tag{13}$$

which is claimed to be accurate to within 1 for $h/w \geq 1$ and $c/w \leq 0.6$. An eccentric through crack in a finite rectangular plate (Figure 3) is analysed next to show the potential of the present method in treating multiple crack tips. For both specimens, h=w=1 are used. As the coefficients in the asymptotic expansions (1) are independent of the material constants, in the computations Young's modulus E is set at 1, and Poisson's ratio at 0.25. The load (Figures 2 and 3) is chosen as =1 with its units consistent with that of E. A state of plane stress is considered and the thickness is assumed to be unity.

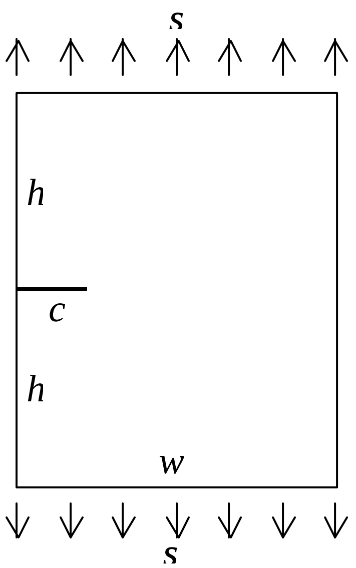

Figure 2: A single edge crack in a finite rectangular plate

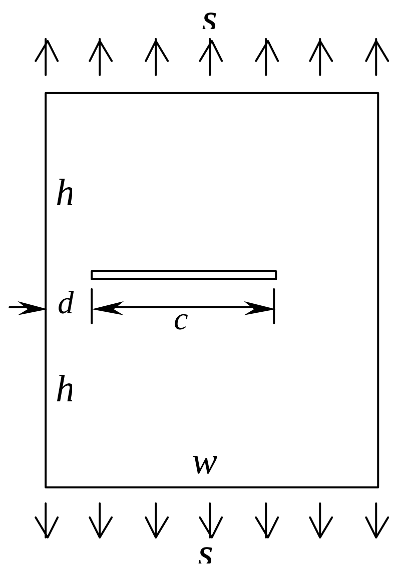

Figure 3: An eccentric through crack in a finite rectangular plate

Only the upper half of the plate in Figure 2 or 3 is considered and divided into 10 10=100 regular elements because of symmetry. The bilinear 4-node isoparametric element is used together with a 2 2 Gauss quadrature. The normal nodal displacements are fixed on the axis of symmetry. r_m=0.1$\sqrt{2}$ is used throughout the computations.

For the SEC with c/w=0.3, the computed a_1 with N=42 and various ngaus, order of Gauss quadrature, is plotted in Figure 4. The results obtained with or without constraining the enriched nodes are included. Obviously, results obtained by constraining the enriched nodes and a reduced integration (ngaus=2) are the most accurate. Quadratures higher than order three give almost identical results. Hence in the following we will constrain the enriched nodes. But the quadrature rule will be tested extensively.

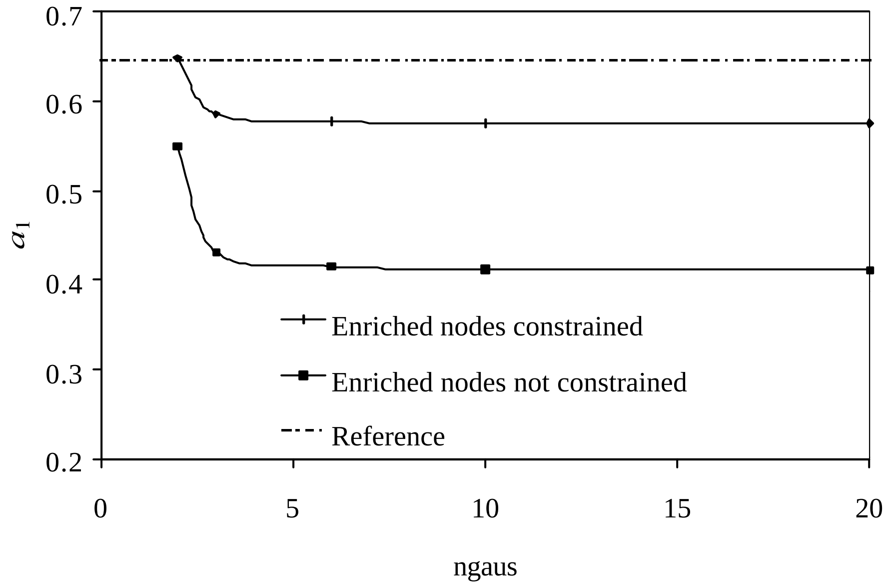

Figure 4: Computed a_1 for various quadrature orders with different constraints on the enriched nodes

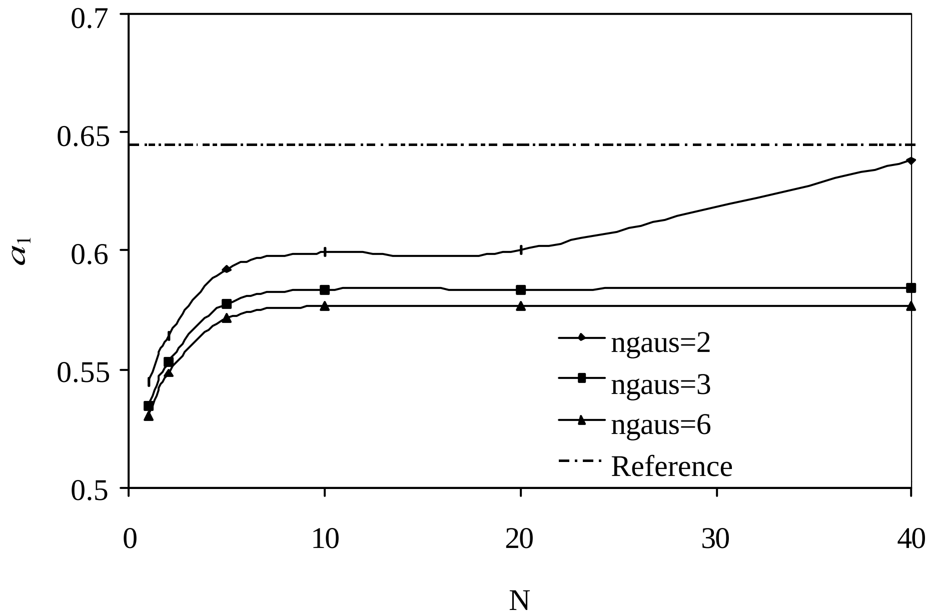

Figure 5: Convergence of the computed a_1 with an increase in the number of retained terms

Again for the SEC with c/w=0.3, the convergence of the computed a_1 with an increase in the retained terms N of various integration orders is studied and reported in Figure 5. It is clear that the second order Gauss integration again gives the most accurate results. Using only the first term or a few higher order terms improves the accuracy but not by much. Desirable accuracy has been maintained by using 40 terms.

By retaining 40-50 terms (a deeper crack needs more terms) and choosing ngaus=2, the computed a_1 for various crack lengths is listed in Table 1 and compared with the solution given by (13). It is clear that very high accuracy (about 1%) has been obtained by the present method.

TABLE 1

COMPUTED a_1 FOR VARIOUS CRACK LENGTHS FOR THE SEC

c/w	0.2	0.3	0.4	0.5	0.6	0.7	0.8
Computed a_1	0.433	0.648	0.936	1.404	2.165	3.554	5.668
(13)	0.434	0.645	0.945	1.421	2.219	3.555	5.731

The eccentric crack in Figure 3 can be easily treated by the present method. d=0.2, c=0.4 and ngaus=2 are used and 43 terms are retained. The computed a_1 for the left and right tips are 0.453 and 0.337, respectively.

The stiffness matrix of an enriched element as well as the system is generally rank deficient. This problem cannot be overcome completely by accurate integration. However, this kind of rank deficiency can be taken care of numerically [8]. In our computations, we used the program given in [10].

DISCUSSION AND CONCLUSIONS

By enriching the FE approximation of the crack tip node as well as its surrounding nodes with not only the first term but also the higher order terms of the crack tip asymptotic field, accurate SIFs are determined directly without extra post-processing. To maintain high accuracy, 40-50 terms in the crack tip asymptotic field should be retained, a reduced quadrature (2 2 Gauss quadrature used in this paper) is desirable, and the enriched nodes should be constrained properly.

Since the general bilinear interpolation cannot improve the approximation of the crack tip field, constraining the enriched nodes improves the condition of rank deficiency and thus the accuracy of the results.

2 2 Gauss quadrature provides the most accurate results because the truncation errors are mainly compensated by the higher order terms, while using accurate integration the errors are averaged among all terms.

Although only Mode I cracks are reported, it is straightforward to employ this method to more complicated crack configurations and/or loading conditions, especially together with the method for incorporating discontinuous displacement fields across the crack face away from the crack tip [5-7].

REFERENCES

1. Tong, P. and Pian, T.H.H. (1973) On the convergence of the finite element method for problems with singularity. *Int. J. Solids Struct.* 9, 313.
2. Liebowitz, H., Moyer, E.T. (1989) Finite element methods in fracture mechanics. *Comput. Struct.* 31, 1.
3. Owen, D.R.J., Fawkes, A.J. (1983). *Engineering fracture mechanics: numerical methods and applications*. Pineridge Press Ltd., Swansea UK.
4. Karihaloo, B.L. and Xiao, Q.Z. (2001) Accurate determination of the coefficients of crack tip asymptotic field by a hybrid element with p-adaptivity. *Engng. Fract. Mech.* (in press).
5. Belytschko, T. and Black, T. (1999) Elastic crack growth in finite elements with minimal remeshing. *Int. J. Numer. Meth. Engng.* 45, 601.
6. Moes, N., Dolbow, J. and Belytschko, T. (1999) A finite element method for crack growth without remeshing. *Int. J. Numer. Meth. Engng.* 46, 131.
7. Daux, C., Moes, N., Dolbow, J., Sukumar, N. and Belytschko, T. (2000) Arbitrary branched and intersecting cracks with the extended finite element method. *Int. J. Numer. Meth. Engng.* 48, 1741.
8. Strouboulis, T., Babuska, I., Copps, K. (2000) The design and analysis of the generalized finite element method. *Comput. Meth. Appl. Mech. Engng.* 181, 43.
9. Rooke, D.P., Cartwright, D.J. (1976). *Compendium of stress intensity factors*. HMSO, London.
10. Bathe, K.J. (1996). *Finite element procedures*. Englewood Cliffs, Prentice-Hall.