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ASYMPTOTIC FIELDS AT A CRACK TIP IN FLAT PLATES

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ABSTRACT

In this paper, we present a new approach to consider the near-tip fields of a crack in elastic flat plate subjected to bending forces. The Reissner assumptions of the plate theory were adopted. By introducing the dual variable vectors in the state space, the governing equations were established in the frame of the Hamiltonian system. Following problems were solved by using the present approach: (1): Cracks in elastic homogeneous plates. (2): Cracks formed by several homogeneous plates. Interface cracks between two dissimilar plates and cracks meeting an interface between two elastic plates are two special cases of this problem. (3): Cracks in orthotropic plates. (4): Cracks formed by several orthotropic plates. This work shows the efficiency and the simplicity of the present theory in studying the crack-tip asymptotic fields in plates.

KEYWORDS

Crack, plate, asymptotic analysis, near-tip fields, orthotropy, interface

1. INTRODUCTION

The principal theories studying the asymptotic fields near a crack tip in a plate loaded by bending forces were established in the 60's of the precedent century (Sih, 1965, Knowels and Wnag, 1960, Hartranft and Sih, 1968, 1970 etc.). Some of them were established on the basis of Poisson-Kirchhoff's thin plate theory, others on the basis of the Ressner theory. The Poisson-Kirchhoff theory provides rather simple mathematical procedures, but gives some physically incorrect behaviors about the near-tip fields. On the other hand, Ressner's thin plate theory gives physically more reasonable results, but the solution of the six-order differential equations remains difficult for some problems posed in this topic.

In this paper, we propose a new approach to find out asymptotic fields near a crack tip in thin plates loaded by bending. By choosing appropriate dual variables in the state space, we can establish the governing equations of the problem in the frame of the Hamiltonian system. All equations found are presented in the form of a system of first-order differential equations. Therefore, one can easily perform the separation of the variables and resolve the corresponding eigenvalue problems. The mathematical approaches are quite simple and a large range of problems in this domain can be dealt with.

2. FUNTAMENTAL EQUATIONS

Consider a semi-infinite crack in a thin elastic plate of thickness h. We adopt the hypothesis made by Reissner about the deformation of thin plates. (1): The strain and stress at the direction normal to the mid-plane are neglected, i.e.:

$$\sigma_z = \varepsilon_z = 0$$
$$w = w(r,\theta) \qquad (1)$$

(2): The in-plane displacements depend linearly on the thickness coordinate z:

$$u_r = z\tilde{u}_r(r,\theta)$$
$$u_\theta = z\tilde{u}_\theta(r,\theta) \qquad (2)$$

Where $\tilde{u}_r$ and $\tilde{u}_\theta$ are functions independent of the z coordinate. We write now the equilibrium equations in the cylindrical coordinate system:

$$\frac{\partial M_{rr}}{\partial r}+\frac{1}{r}\frac{\partial M_{r\theta}}{\partial \theta}+\frac{M_{rr}-M_{\theta\theta}}{r}=Q_{rz} \quad \frac{\partial M_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial M_{\theta\theta}}{\partial \theta}+\frac{2M_{r\theta}}{r}=Q_{\theta z} \quad \frac{\partial Q_{rz}}{\partial r}+\frac{1}{r}\frac{\partial Q_{\theta z}}{\partial \theta}+\frac{Q_{rz}}{r}=-q \qquad (3)$$

The relationships between the strain and displacement components are, according to assumption (2):

$$\varepsilon_r = z\frac{\partial \tilde{u}_r}{\partial r} \quad \varepsilon_\theta = \frac{z}{r}\left(\frac{\partial \tilde{u}_\theta}{\partial \theta}+\tilde{u}_r\right) \quad 2\varepsilon_{r\theta} = z\frac{\partial \tilde{u}_\theta}{\partial r}+\frac{z}{r}\left(\frac{\partial \tilde{u}_r}{\partial \theta}-\tilde{u}_\theta\right) \quad 2\varepsilon_{rz} = \tilde{u}_r+\frac{\partial w}{\partial r} \quad 2\varepsilon_{\theta z} = \tilde{u}_\theta+\frac{1}{r}\frac{\partial w}{\partial \theta} \qquad (4)$$

According to the Hooke law, we can directly write the relationships between the displacement and stress components:

$$z\frac{\partial \tilde{u}_r}{\partial r}=\frac{1}{E}(\sigma_r-\nu\sigma_\theta) \quad \frac{z}{r}\left(\frac{\partial \tilde{u}_\theta}{\partial \theta}+\tilde{u}_r\right)=\frac{1}{E}(\sigma_\theta-\nu\sigma_r) \quad z\frac{\partial \tilde{u}_\theta}{\partial r}+\frac{z}{r}\left(\frac{\partial \tilde{u}_r}{\partial \theta}-\tilde{u}_\theta\right)=\frac{2(1+\nu)}{E}\sigma_{r\theta}$$
$$\tilde{u}_r+\frac{\partial w}{\partial r}=\frac{2(1+\nu)}{E}\sigma_{rz} \quad \tilde{u}_\theta+\frac{1}{r}\frac{\partial w}{\partial \theta}=\frac{2(1+\nu)}{E}\sigma_{\theta z} \qquad (5)$$

We multiply the three first equations by z then perform integration through the thickness. For the two last equations, we just perform integration. We obtain:

$$\frac{\partial \tilde{u}_r}{\partial r}=\frac{12}{Eh^3}(M_{rr}-\nu M_{\theta\theta}) \quad \frac{1}{r}\left(\frac{\partial \tilde{u}_\theta}{\partial \theta}+\tilde{u}_r\right)=\frac{12}{Eh^3}(M_{\theta\theta}-\nu M_{rr}) \quad \frac{\partial \tilde{u}_\theta}{\partial r}+\frac{1}{r}\left(\frac{\partial \tilde{u}_r}{\partial \theta}-\tilde{u}_\theta\right)=\frac{24(1+\nu)}{Eh^3}M_{r\theta}$$
$$\tilde{u}_r+\frac{\partial w}{\partial r}=\frac{2(1+\nu)}{Ehk}Q_{rz} \quad \tilde{u}_\theta+\frac{1}{r}\frac{\partial w}{\partial \theta}=\frac{2(1+\nu)}{Ehk}Q_{\theta z} \qquad (6)$$

where $k = 5/6$ is a corrector constant in order to take the parabolic distribution of the shear stresses into account. Equation (3) and (6) are the fundamental equations we use in this work. The boundary conditions at the crack lips are written as following:

$$M_{r\theta}(\theta=\pm\pi)=M_{\theta\theta}(\theta=\pm\pi)=Q_{\theta z}(\theta=\pm\pi)=0 \qquad (7)$$

In order to solve these fundamental equations, we perform the following variable changes:

$$r=e^\xi \quad w=r\tilde{w} \quad M_{rr}=\frac{\tilde{M}_{rr}}{r} \quad M_{\theta\theta}=\frac{\tilde{M}_{\theta\theta}}{r} \quad M_{r\theta}=\frac{\tilde{M}_{r\theta}}{r} \qquad (8)$$

Then equations (3) and (6) become respectively:

$$\frac{\partial \tilde{M}_{rr}}{\partial \xi}+\frac{\partial \tilde{M}_{r\theta}}{\partial \theta}-\tilde{M}_{\theta\theta}=e^{2\xi}Q_{rz} \quad \frac{\partial \tilde{M}_{r\theta}}{\partial \xi}+\frac{\partial \tilde{M}_{\theta\theta}}{\partial \theta}+\tilde{M}_{r\theta}=e^{2\xi}Q_{\theta z} \quad \frac{\partial Q_{rz}}{\partial \xi}+\frac{\partial Q_{\theta z}}{\partial \theta}+Q_{rz}=-e^\xi q \qquad (9)$$

$$\frac{\partial \tilde{u}_r}{\partial \xi}=\frac{12}{Eh^3}(\tilde{M}_{rr}-\nu\tilde{M}_{\theta\theta}) \quad \frac{\partial \tilde{u}_\theta}{\partial \theta}+\tilde{u}_r=\frac{12}{Eh^3}(\tilde{M}_{\theta\theta}-\nu\tilde{M}_{rr}) \quad \frac{\partial \tilde{u}_\theta}{\partial \xi}+\frac{\partial \tilde{u}_r}{\partial \theta}-\tilde{u}_\theta=\frac{24(1+\nu)}{Eh^3}\tilde{M}_{r\theta}$$
$$\tilde{u}_r+\tilde{w}+\frac{\partial \tilde{w}}{\partial \xi}=\frac{2(1+\nu)}{Ehk}Q_{rz} \quad \tilde{u}_\theta+\frac{\partial w}{\partial \theta}=\frac{2(1+\nu)}{Ehk}Q_{\theta z} \qquad (10)$$

3. TRANSFORMATION INTO THE HAMILTONIAN SYSTEM ON THE BASIS OF THE RADIAL COORDINATE

In this case, we note $\frac{\partial}{\partial \xi}=(\dot{\ })$, and then we define the dual variables as following:

$$\boldsymbol{v}=\{\boldsymbol{v}_1^T \quad \boldsymbol{v}_2^T\}^T \quad \boldsymbol{v}_1=\{\boldsymbol{q}_1^T \quad \boldsymbol{p}_1^T\}^T \quad \boldsymbol{v}_2=\{\boldsymbol{q}_2^T \quad \boldsymbol{p}_2^T\}^T$$
$$\boldsymbol{q}_1=\{\tilde{u}_r \quad \tilde{u}_\theta\}^T \quad \boldsymbol{q}_2=\tilde{w} \quad \boldsymbol{p}_1=\{\tilde{M}_{rr} \quad \tilde{M}_{r\theta}\}^T \quad \boldsymbol{p}_2=Q_{rz} \qquad (11)$$

We eliminate from (10) the quantities that do not exist in the above dual variables, namely:

$$Q_{\theta z}=\frac{Ehk}{2(1+\nu)}\left(\tilde{u}_\theta+\frac{\partial\tilde{w}}{\partial\theta}\right)\quad \tilde{M}_{\theta\theta}=\frac{Eh^3}{12}\left(\frac{\partial\tilde{u}_\theta}{\partial\theta}+\tilde{u}_r\right)+\nu\tilde{M}_{rr}\qquad(12)$$

By neglecting the terms of higher orders as $r\to 0$, we obtain the following dual differential equations:

$$\dot{\boldsymbol{v}}=\boldsymbol{H}\boldsymbol{v}\qquad(13)$$

with:

$$\boldsymbol{H}=\begin{bmatrix}\boldsymbol{H}_{11} & \boldsymbol{0}\\ \boldsymbol{H}_{21} & \boldsymbol{H}_{22}\end{bmatrix}\qquad \begin{array}{l}\dot{\boldsymbol{v}}_1=\boldsymbol{H}_{11}\boldsymbol{v}_1\\ \dot{\boldsymbol{v}}_2=\boldsymbol{H}_{22}\boldsymbol{v}_2+\boldsymbol{H}_{21}\boldsymbol{v}_1\end{array}\qquad(14)$$

$$\boldsymbol{H}_{11}=\begin{bmatrix}-\nu & -\nu\frac{\partial}{\partial\theta} & \frac{12(1-\nu^2)}{Eh^3} & 0\\ -\frac{\partial}{\partial\theta} & 1 & 0 & \frac{24(1+\nu)}{Eh^3}\\ \frac{Eh^3}{12} & \frac{Eh^3}{12}\frac{\partial}{\partial\theta} & \nu & -\frac{\partial}{\partial\theta}\\ -\frac{Eh^3}{12}\frac{\partial}{\partial\theta} & -\frac{Eh^3}{12}\frac{\partial^2}{\partial\theta^2} & -\nu\frac{\partial}{\partial\theta} & -1\end{bmatrix}\qquad(15)$$

$$\boldsymbol{H}_{22}=\begin{bmatrix}-1 & \frac{2(1+\nu)}{Ehk}\\ -\frac{Ehk}{2(1+\nu)}\frac{\partial^2}{\partial\theta^2} & -1\end{bmatrix}\qquad \boldsymbol{H}_{21}=\begin{bmatrix}-1 & 0 & 0 & 0\\ 0 & -\frac{Ehk}{2(1+\nu)}\frac{\partial}{\partial\theta} & 0 & 0\end{bmatrix}$$

The resolution of the governing equation (13) with the boundary conditions (7) is easy. We first write the solution under separable form:

$$\boldsymbol{v}=e^{\mu\xi}\psi(\theta)\qquad(16)$$

where μ is an eigenvalue. $\psi(\theta)$ is the corresponding eigenvector. Substituting (16) into (13) gives:

$$(\boldsymbol{H}-\boldsymbol{I}\mu)\psi(\theta)=\boldsymbol{0}\qquad(17)$$

From (14), we remark that the solution of $\boldsymbol{v}_1$ is independent of $\boldsymbol{v}_2$. So we can first solve the eigenvalue problem (17) for $\boldsymbol{v}_1$. We can easily find the eigenvalues $\mu=0,\pm 1/2,1,3/2,\ldots$. In crack problems, only eigenvectors of $\boldsymbol{v}_1$ for positive eigenvalues exist. The singular fields for $\boldsymbol{v}_1$ can therefore easily be obtained, namely:

$$\begin{aligned}
u_r&=\frac{K_1 z}{2D}\sqrt{\frac{r}{2\pi}}\left[-\cos\frac{3\theta}{2}+(2\chi-1)\cos\frac{\theta}{2}\right]+\frac{K_2 z}{2D}\sqrt{\frac{r}{2\pi}}\left[3\sin\frac{3\theta}{2}-(2\chi-1)\sin\frac{\theta}{2}\right]\\
u_\theta&=\frac{K_1 z}{2D}\sqrt{\frac{r}{2\pi}}\left[\sin\frac{3\theta}{2}-(2\chi+1)\sin\frac{\theta}{2}\right]+\frac{K_2 z}{2D}\sqrt{\frac{r}{2\pi}}\left[3\cos\frac{3\theta}{2}-(2\chi+1)\cos\frac{\theta}{2}\right]\\
M_{rr}&=\frac{K_1}{4\sqrt{2\pi r}}\left[-\cos\frac{3\theta}{2}+5\cos\frac{\theta}{2}\right]+\frac{K_2}{4\sqrt{2\pi r}}\left[3\sin\frac{3\theta}{2}-5\sin\frac{\theta}{2}\right]\qquad(18)\\
M_{\theta\theta}&=\frac{K_1}{4\sqrt{2\pi r}}\left[\cos\frac{3\theta}{2}+3\cos\frac{\theta}{2}\right]+\frac{K_2}{4\sqrt{2\pi r}}\left[-3\sin\frac{3\theta}{2}-3\sin\frac{\theta}{2}\right]\\
M_{r\theta}&=\frac{K_1}{4\sqrt{2\pi r}}\left[\sin\frac{3\theta}{2}+\sin\frac{\theta}{2}\right]+\frac{K_2}{4\sqrt{2\pi r}}\left[3\cos\frac{3\theta}{2}+\cos\frac{\theta}{2}\right]
\end{aligned}$$

where K_1 and K_2 are stress intensity factors, $D=Eh/12(1+\nu)$. From dimension analysis, we know that the eigenvalues for $\boldsymbol{v}_2$ may be negative, and the most negative eigenvalue is $\mu=-1/2$. Since the eigenvector of $\boldsymbol{v}_1$ for $\mu=-1/2$ is nil, from (14), we have:

$$\dot{\boldsymbol{v}}_2=\boldsymbol{H}_{22}\boldsymbol{v}_2\qquad(19)$$

The solution of (19) with the boundary solution (7) is:

$$w=\frac{h^2}{3D}\sqrt{\frac{r}{2\pi}}K_3\sin\frac{\theta}{2}\quad Q_{rz}=\frac{K_3}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\quad Q_{\theta z}=\frac{K_3}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\qquad(20)$$

(18) and (20) are just the solution found out by Hartranft and Sih (1968) by using an integral transform technique. Here we find it with rather a simple approach.

4. TRANSFORMATION INTO THE HAMILTONIAN SYSTEM ON THE BASIS OF THE ANGULAR COORDINATE, MULTI-MATERIAL PROBLEMS

If we define the dual variables as follows:

$$\boldsymbol{v} = \left\{\boldsymbol{v}_1^T \quad \boldsymbol{v}_2^T\right\}^T \quad \boldsymbol{v}_1 = \left\{\boldsymbol{q}_1^T \quad \boldsymbol{p}_1^T\right\}^T \quad \boldsymbol{v}_2 = \left\{\boldsymbol{q}_2^T \quad \boldsymbol{p}_2^T\right\}^T$$
$$\boldsymbol{q}_1 = \left\{\tilde{u}_r \quad \tilde{u}_\theta\right\}^T \quad \boldsymbol{q}_2 = \tilde{w} \quad \boldsymbol{p}_1 = \left\{\tilde{M}_{r\theta} \quad \tilde{M}_{\theta\theta}\right\}^T \quad \boldsymbol{p}_2 = Q_{\theta z} \tag{21}$$

and we note $\frac{\partial}{\partial\theta} = \left(^\bullet\right)$. We eliminate from (10) quantities that don't exist in the dual variables defined above, namely:

$$\tilde{M}_{rr} = \frac{Eh^3}{12}\frac{\partial \tilde{u}_r}{\partial \xi} + \nu \tilde{M}_{\theta\theta} \quad Q_{rz} = \frac{Ehk}{2(1+\nu)}\left(\tilde{u} + \tilde{w} + \frac{\partial \tilde{w}}{\partial \xi}\right) \tag{22}$$

We obtain another dual differential equations:

$$\dot{\boldsymbol{v}} = \boldsymbol{H}\boldsymbol{v} \tag{23}$$

with

$$\boldsymbol{H} = \begin{bmatrix} \boldsymbol{H}_{11} & \boldsymbol{0} \\ \boldsymbol{H}_{21} & \boldsymbol{H}_{22} \end{bmatrix} \quad \begin{matrix} \dot{\boldsymbol{v}}_1 = \boldsymbol{H}_{11}\boldsymbol{v}_1 \\ \dot{\boldsymbol{v}}_2 = \boldsymbol{H}_{22}\boldsymbol{v}_2 + \boldsymbol{H}_{21}\boldsymbol{v}_1 \end{matrix} \tag{24}$$

$$\boldsymbol{H}_{11} = \begin{bmatrix} 0 & 1 - \frac{\partial}{\partial\xi} & \frac{24(1+\nu)}{Eh^3} & 0 \\ -1 - \nu\frac{\partial}{\partial\xi} & 0 & 0 & \frac{12(1-\nu^2)}{Eh^3} \\ -\frac{Eh^3}{12}\frac{\partial^2}{\partial\xi^2} & 0 & 0 & 1 - \nu\frac{\partial}{\partial\xi} \\ 0 & 0 & -1 - \frac{\partial}{\partial\xi} & 0 \end{bmatrix} \tag{25}$$
$$\boldsymbol{H}_{22} = \begin{bmatrix} 0 & \frac{2(1+\nu)}{Ehk} \\ -\frac{Ehk}{2(1+\nu)}\left(\frac{\partial}{\partial\xi} + 1\right)^2 & 0 \end{bmatrix} \quad \boldsymbol{H}_{21} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -\frac{Ehk}{2(1+\nu)}\left(\frac{\partial}{\partial\xi} + 1\right) & 0 & 0 & 0 \end{bmatrix}$$

The solution of (23) gives the same results as those found in the precedent section.
The main advantage of this approach is its high capacity to deal with the multi-material problems. Imagine a crack or a notch formed by *n* homogenous plates, all interfaces between two of these plates intercept at the crack tip. The boundary conditions at the crack lip are therefore:

$$\boldsymbol{p}^{(1)}(\theta = \theta_0 = -\pi) = \boldsymbol{p}^{(n)}(\theta = \theta_n = \pi) = \boldsymbol{0} \quad \boldsymbol{p} = \left\{\boldsymbol{p}_1^T \quad \boldsymbol{p}_2^T\right\}^T \tag{26}$$

and the continuity conditions across the interfaces are:

$$\boldsymbol{v}^{(i)}(\theta = \theta_i) = \boldsymbol{v}^{(i+1)}(\theta = \theta_i) \tag{27}$$

where the superscript $^{(i)}$ indicates the quantities in the zone occupied by the plate *i*. It is seen that the variable vector $\boldsymbol{v}$ is continuous across all the interfaces. This makes the solution of the multi-material problems much easier. In each zone, we can establish the governing differential equation (23), namely:

$$\dot{\boldsymbol{v}}^{(i)} = \boldsymbol{H}^{(i)}\boldsymbol{v}^{(i)} \tag{28}$$

We look for only the eigenvalues leading to singular stress filed near the crack tip. According to the analysis made in the precedent section, the vectors $\boldsymbol{v}_1$ corresponding to negative eigenvectors are nil, while a singular vector $\boldsymbol{v}_2$ requires negative eigenvalues. Therefore, we can divide (28) into two distinguish equations:

$$\dot{\boldsymbol{v}}_1^{(i)} = \boldsymbol{H}_{11}^{(i)}\boldsymbol{v}_1^{(i)} \quad \dot{\boldsymbol{v}}_2^{(i)} = \boldsymbol{H}_{22}^{(i)}\boldsymbol{v}_2^{(i)} \tag{29}$$

We can resolve (29) by writing $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ under separable form:

$$\boldsymbol{v}_1^{(i)} = e^{\mu_1\xi}\psi_1^{(i)}(\theta) \quad \boldsymbol{v}_2^{(i)} = e^{\mu_2\xi}\psi_2^{(i)}(\theta) \tag{30}$$

Substituting (30) into (29) gives:

$$\dot{\psi}_1^{(i)} = \boldsymbol{H}_{11}^{(i)}(\mu_1)\psi_1^{(i)} \quad \dot{\psi}_1^{(i)} = \boldsymbol{H}_{22}^{(i)}(\mu_2)\psi_1^{(i)} \tag{31}$$

where

$$\boldsymbol{H}_{11}^{(i)}(\mu_1) = \begin{bmatrix} 0 & 1-\mu_1 & \dfrac{24(1+\nu^{(i)})}{E^{(i)}h^3} & 0 \\ -1-\nu^{(i)}\mu_1 & 0 & 0 & \dfrac{12(1-\nu^{(i)2})}{E^{(i)}h^3} \\ -\dfrac{E^{(i)}h^3}{12}\mu_1^2 & 0 & 0 & 1-\nu^{(i)}\mu_1 \\ 0 & 0 & -1-\mu_1 & 0 \end{bmatrix} \quad \boldsymbol{H}_{22}^{(i)}(\mu_2) = \begin{bmatrix} 0 & \dfrac{2(1+\nu^{(i)})}{E^{(i)}hk} \\ -\dfrac{E^{(i)}hk}{2(1+\nu^{(i)})}(\mu_2+1)^2 & 0 \end{bmatrix} \tag{32}$$

The solution of (31) is immediately written as follows:

$$\psi_1^{(i)}(\theta) = e^{\boldsymbol{H}_{11}^{(i)}(\theta-\theta_{i-1})}\psi_1^{(i)}(\theta_{i-1}) \quad \psi_2^{(i)}(\theta) = e^{\boldsymbol{H}_{22}^{(i)}(\theta-\theta_{i-1})}\psi_2^{(i)}(\theta_{i-1}) \tag{33}$$

According to the continuity conditions (27), we obtain the relationship between the ψ's at the two crack lips:

$$\psi_1^{(n)}(\theta=\pi) = \boldsymbol{G}^1\psi_1^{(0)}(\theta=-\pi) \quad \psi_2^{(n)}(\theta=\pi) = \boldsymbol{G}^2\psi_2^{(0)}(\theta=-\pi) \tag{34}$$

with:

$$\boldsymbol{G}^1 = \prod_{i=n}^{1} e^{\boldsymbol{H}_{11}^{(i)}(\theta_i-\theta_{i-1})} \quad \boldsymbol{G}^2 = \prod_{i=n}^{1} e^{\boldsymbol{H}_{22}^{(i)}(\theta_i-\theta_{i-1})} \tag{35}$$

where $\boldsymbol{G}^1$ is a 4×4 matrix and $\boldsymbol{G}^2$ is a 2×2 matrix. According to the boundary conditions (26), we have finally the following conditions allowing calculation of the eigenvalues:

$$\det\begin{bmatrix} \boldsymbol{G}_{31}^1 & \boldsymbol{G}_{32}^1 \\ \boldsymbol{G}_{41}^1 & \boldsymbol{G}_{42}^1 \end{bmatrix} = 0 \quad \boldsymbol{G}_{21}^2 = 0 \tag{36}$$

Once the eigenvalues obtained, the corresponding eigenvectors can immediately be computed from (33).

5. ORTHOTROPIC PLATES

Anisotropy is a very important quality in composite plates. Now let us consider an orthotropic plate that is habitually used in engineering applications. If the mid-plane is perpendicular to the orthotropic axis, one can write the relationship between the stress and strain components in the cylindrical coordinate system:

$$\sigma = \boldsymbol{C}\varepsilon$$

$$\sigma = \{\sigma_{rr} \quad \sigma_{\theta\theta} \quad \sigma_{zz} \quad \sigma_{r\theta} \quad \sigma_{rz} \quad \sigma_{\theta z}\}^T \quad \varepsilon = \{\varepsilon_{rr} \quad \varepsilon_{\theta\theta} \quad \varepsilon_{zz} \quad 2\varepsilon_{r\theta} \quad 2\varepsilon_{rz} \quad 2\varepsilon_{\theta z}\}^T \tag{37}$$

$$\boldsymbol{C} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & 0 & 0 \\ c_{21} & c_{22} & c_{23} & c_{24} & 0 & 0 \\ c_{31} & c_{32} & c_{33} & c_{34} & 0 & 0 \\ c_{41} & c_{42} & c_{44} & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & c_{56} \\ 0 & 0 & 0 & 0 & c_{65} & c_{66} \end{bmatrix}$$

The components of the stiffness matrix may be function of θ. From (37), one can easily find:

$$\begin{Bmatrix} \widetilde{M}_{rr} \\ \widetilde{M}_{\theta\theta} \\ \widetilde{M}_{r\theta} \\ Q_{rz} \\ Q_{\theta z} \end{Bmatrix} = \begin{bmatrix} \widetilde{c}_{11} & \widetilde{c}_{12} & \widetilde{c}_{14} & 0 & 0 \\ \widetilde{c}_{21} & \widetilde{c}_{22} & \widetilde{c}_{24} & 0 & 0 \\ \widetilde{c}_{41} & \widetilde{c}_{42} & \widetilde{c}_{44} & 0 & 0 \\ 0 & 0 & 0 & \widetilde{c}_{55} & \widetilde{c}_{56} \\ 0 & 0 & 0 & \widetilde{c}_{65} & \widetilde{c}_{66} \end{bmatrix} \begin{bmatrix} \partial/\partial\xi & 0 & 0 \\ 1 & \partial/\partial\theta & 0 \\ \partial/\partial\theta & 1+\partial/\partial\xi & 0 \\ 1 & 0 & \partial/\partial\xi \\ 0 & 1 & \partial/\partial\theta \end{bmatrix} \begin{Bmatrix} \widetilde{u}_r \\ \widetilde{u}_\theta \\ \widetilde{w} \end{Bmatrix} \quad \begin{matrix} \widetilde{c}_{ij} = \dfrac{h^3}{12}c_{ij} & i,j=1,2,4 \\ \widetilde{c}_{ij} = khc_{ij} & i,j=5,6 \end{matrix} \tag{38}$$

By choosing the following dual variable vectors:

$$\boldsymbol{v} = \{\boldsymbol{q}^T \quad \boldsymbol{p}^T\} \quad \boldsymbol{q} = \{\widetilde{u}_r \quad \widetilde{u}_\theta \quad \widetilde{w}\}^T \quad \boldsymbol{p} = \{\widetilde{M}_{r\theta} \quad \widetilde{M}_{\theta\theta} \quad Q_{\theta z}\}^T \tag{39}$$

and by neglecting the high order quantities as $r \to 0$, we can find the following dual differential equations:

$$\dot{\boldsymbol{v}} = \boldsymbol{H}\boldsymbol{v}$$

$$\boldsymbol{H} = \begin{bmatrix} -\boldsymbol{C}_d^{-1}\boldsymbol{C}_{d1} & \boldsymbol{C}_d^{-1} \\ \boldsymbol{C}_{f1}\left(\boldsymbol{C}_{e1} - \boldsymbol{C}_e\boldsymbol{C}_d^{-1}\boldsymbol{C}_{d1}\right) & \boldsymbol{C}_f + \boldsymbol{C}_{f1}\boldsymbol{C}_e\boldsymbol{C}_d^{-1} \end{bmatrix} \qquad (40)$$

with $\dfrac{\partial}{\partial\theta} = (\bullet)$ and

$$\boldsymbol{C}_{d1} = \begin{bmatrix} \tilde{c}_{41}\dfrac{\partial}{\partial\xi} + \tilde{c}_{42} & \tilde{c}_{44}\left(\dfrac{\partial}{\partial\xi} + 1\right) & 0 \\ \tilde{c}_{21}\dfrac{\partial}{\partial\xi} + \tilde{c}_{22} & \tilde{c}_{24}\left(\dfrac{\partial}{\partial\xi} + 1\right) & 0 \\ \tilde{c}_{65} & \tilde{c}_{66} & \tilde{c}_{65}\dfrac{\partial}{\partial\xi} \end{bmatrix} \quad \boldsymbol{C}_d = \begin{bmatrix} \tilde{c}_{44} & \tilde{c}_{42} & 0 \\ \tilde{c}_{24} & \tilde{c}_{22} & 0 \\ 0 & 0 & \tilde{c}_{66} \end{bmatrix}$$

$$\boldsymbol{C}_{e1} = \begin{bmatrix} \tilde{c}_{11}\dfrac{\partial}{\partial\xi} + \tilde{c}_{12} & \tilde{c}_{14}\left(\dfrac{\partial}{\partial\xi} + 1\right) & 0 \\ \tilde{c}_{55} & \tilde{c}_{56} & \tilde{c}_{55}\dfrac{\partial}{\partial\xi} \end{bmatrix} \quad \boldsymbol{C}_e = \begin{bmatrix} \tilde{c}_{14} & \tilde{c}_{12} & 0 \\ 0 & 0 & \tilde{c}_{56} \end{bmatrix} \qquad (41)$$

$$\boldsymbol{C}_f = \begin{bmatrix} 0 & 1 & 0 \\ -\left(1 + \dfrac{\partial}{\partial\xi}\right) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \boldsymbol{C}_{f1} = \begin{bmatrix} -\dfrac{\partial}{\partial\xi} & 0 \\ 0 & 0 \\ 0 & -\left(1 + \dfrac{\partial}{\partial\xi}\right) \end{bmatrix}$$

Here again, we establish the standard form of the governing equation (40) in the Hamiltonian system. We can then perform the separation of the variables and solve the corresponding eigenvalue problem as described in the precedent section. As for the isotropic materials, cracks or notches formed by several anisotrapic plates can also be dealt with in a very similar manner.

6. CONCLUSIONS

In this paper, we have developed a new approach to deal with asymptotic fields near a crack tip in thin plates subjected to bending forces. Ressner hypothesis are used in this theory. By establishing dual differential equations in the frame of the Hamiltonian system, a large range of problems in this topic, some of them are often difficult to treat with the traditional techniques, can be solved in rather a simple way.

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