ANALYSIS OF A PENNY-SHAPED CRACK IN A TRANSVERSELY ISOTROPIC PIEZOELECTRIC MEDIUM

Ming-Hao Zhao, Pin Tong and Tong-Yi Zhang*

Department of Mechanical Engineering, Hong Kong University of Science and Technology
Clear Water Bay, Kowloon, Hong Kong, China, *E-mail: mezhangt@ust.hk

ABSTRACT

The present work analytically studies an ellipsoidal cavity and then a penny-shaped crack in a transversely isotropic piezoelectric medium under uniform remote mechanical and electrical loading. Three-dimensional (3D) analytic solutions are derived for the mechanical and electrical fields in the piezoelectric medium and for the electric field within the cavity. An effective dielectric constant of the material is introduced here, which involves the material dielectric, piezoelectric and elastic constants. The results indicate that the electric field within the cavity is uniform and its magnitude increases with decreasing the ratio β^* of the dielectric constant of the cavity to the 3D effective dielectric constant of the material. When the cavity is reduced into a penny-shaped crack, the crack mechanical and electrical fields depend on the ratio of α/β^*, where α is the ratio of the minor semi-axis to the major semi-axis of the ellipsoidal cavity. The electrically impermeable and permeable penny-shaped cracks are just two extreme cases of the present solutions, corresponding to $\alpha/\beta^*\to\infty$ and $\alpha/\beta^*\to 0$, respectively.

KEYWORDS

piezoelectric medium, penny-shaped crack, intensity factor, analytic solution

INTRODUCTION

In purely elastic fracture mechanics, a crack is usually treated as a mathematical slit without any thickness. To utilize this simplification for electrically insulating cracks in piezoelectric materials, one has to assume an electrically insulating crack to be electrically impermeable [1-10] or permeable [11, 12]. However, a real crack has a finite nonzero width and the crack geometry has a great influence on the fracture behavior of the materials [13]. Furthermore, the two-dimensional (2D) results show that the crack fields depend on the ratio of α/β [14, 15], where β is the ratio of the dielectric constant of the cavity (or crack) to the 2D effective dielectric constant of the material. The electrically impermeable and permeable boundary conditions along the crack faces are only two extreme cases, corresponding to $\alpha/\beta\to\infty$ and $\alpha/\beta\to 0$, respectively. In this paper, we will demonstrate the similar results for penny-shaped cracks.

BASIC EQUATIONS

In three-dimensional piezoelectric elasticity, the equilibrium equations, in terms of stress σ_{ij} and electric displacement D_i, are given by

$$\sigma_{ij,j} = 0, \quad D_{i,i} = 0, \qquad i, j=1, 2, 3. \tag{1}$$

The kinematic equations read

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \quad E_i = -\phi_{,i}, \qquad i, j=1, 2, 3, \tag{2}$$

where ϕ, u_i, E_i and ε_{ij} denote the electric potential, displacement vector, electric field vector and the strain tensor, respectively. The constitutive equations take the form

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl} - e_{kij}E_k, \; D_k = e_{kij}\varepsilon_{ij} + \kappa_{kl}E_l, \quad i, j, k, l=1, 2, 3, \tag{3}$$

where c_{ijkl}, e_{kij} and κ_{kl} are the elastic, piezoelectric and dielectric constants, respectively.
Let the $r\theta$ - plane of the cylindrical coordinate system (r, θ, z) coincide with the isotropic plane of the transversely isotropic medium and the poling direction be along the z-axis. The displacements and the electric potential may be expressed by the four potential functions U_i (i=1, 2, 3, 4) [7, 11, 16, 17]

$$u_r = \sum_{i=1}^{3}\frac{\partial U_i}{\partial r} - \frac{1}{r}\frac{\partial U_4}{\partial \theta}, \quad u_\theta = \sum_{i=1}^{3}\frac{1}{r}\frac{\partial U_i}{\partial \theta} + \frac{\partial U_4}{\partial r}, \quad u_z = \sum_{i=1}^{3}k_{1i}\frac{\partial U_i}{\partial z}, \quad \varphi = \sum_{i=1}^{3}k_{2i}\frac{\partial U_i}{\partial z}, \tag{4}$$

where k_1 and k_2 are constants to be determined. Putting Eq. (4) into Eqs. (2), (3), and then (1) yields

$$\frac{\partial^2 U_i}{\partial r^2} + \frac{1}{r}\frac{\partial U_i}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U_i}{\partial \theta^2} + \frac{\partial^2 U_i}{\partial z_i^2} = 0, \quad i=1, 2, 3, 4, \tag{5}$$

where $z_i = s_i z$ and $s_i = 1/\sqrt{\lambda_i}$, and

$$\lambda_4 = \frac{2c_{44}}{c_{11} - c_{12}}, \tag{6}$$

and the other three roots λ_i (i=1, 2 3) are determined from the characteristic equation

$$A\lambda^3 + B\lambda^2 + C\lambda + D = 0. \tag{7}$$

In Eq. (7), the constants A, B, C, and D are combinations of material constants and given by

$$\begin{aligned}
&A = c_{11}(e_{15}^2 + c_{44}\kappa_{11}),\\
&B = 2e_{15}(e_{15}c_{13} - e_{33}c_{11}) + 2c_{13}(e_{15}e_{31} + c_{44}\kappa_{11}) - c_{44}(e_{31}^2 + c_{11}\kappa_{33}) + \kappa_{11}(c_{13}^2 - c_{11}c_{33}),\\
&C = e_{33}(e_{33}c_{11} + 2e_{15}c_{44}) - 2e_{33}(e_{15} + e_{31})(c_{13} + c_{44}) - \kappa_{33}(c_{13}^2 + 2c_{13}c_{44} - c_{11}c_{33}) + \kappa_{11}c_{33}c_{44} + c_{33}(e_{15} + e_{31})^2,\\
&D = -c_{44}(e_{33}^2 + c_{33}\kappa_{33}).
\end{aligned} \tag{8}$$

In Eq. (4) k_{1i} and k_{2i} (i=1, 2, 3) are constants related to λ_i by

$$\frac{c_{44} + (c_{13} + c_{44})k_{1i} + (e_{15} + e_{31})k_{2i}}{c_{11}} = \frac{c_{33}k_{1i} + e_{33}k_{2i}}{c_{13} + c_{44} + c_{44}k_{1i} + e_{15}k_{2i}} = \frac{e_{33}k_{1i} - \kappa_{33}k_{2i}}{e_{15} + e_{31} + e_{15}k_{1i} - \kappa_{11}k_{2i}} = \lambda_i. \tag{9}$$

Finally, we express the stresses and the electric displacements in terms of the potential functions

$$
\begin{aligned}
\sigma_{rr} &= \sum_{i=1}^{3}\left[c_{11}\frac{\partial^2 U_i}{\partial r^2} + c_{12}\frac{1}{r}\frac{\partial U_i}{\partial r} + c_{12}\frac{1}{r^2}\frac{\partial^2 U_i}{\partial \theta^2} + (c_{13}k_{1i} + e_{31}k_{2i})\frac{\partial^2 U_i}{\partial z^2}\right] + (c_{11} - c_{12})\left(\frac{1}{r^2}\frac{\partial U_4}{\partial \theta} - \frac{1}{r}\frac{\partial^2 U_4}{\partial \theta \partial r}\right), \\
\sigma_{\theta\theta} &= \sum_{i=1}^{3}\left[c_{12}\frac{\partial^2 U_i}{\partial r^2} + c_{11}\frac{1}{r}\frac{\partial U_i}{\partial r} + c_{11}\frac{1}{r^2}\frac{\partial^2 U_i}{\partial \theta^2} + (c_{13}k_{1i} + e_{31}k_{2i})\frac{\partial^2 U_i}{\partial z^2}\right] - (c_{11} - c_{12})\left(\frac{1}{r^2}\frac{\partial U_4}{\partial \theta} - \frac{1}{r}\frac{\partial^2 U_4}{\partial \theta \partial r}\right), \\
\sigma_{zz} &= \sum_{i=1}^{3}\lambda_i (c_{44} + c_{144}k_{1i} + e_{15}k_{2i})\frac{\partial^2 U_i}{\partial z^2}, \\
\sigma_{z\theta} &= \sum_{i=1}^{3}\left[(c_{44} + c_{44}k_{1i} + e_{15}k_{2i})\frac{1}{r}\frac{\partial^2 U_i}{\partial z \partial \theta}\right] + c_{44}\frac{\partial^2 U_4}{\partial r \partial z}, \\
\sigma_{zr} &= \sum_{i=1}^{3}\left[(c_{44} + c_{44}k_{1i} + e_{15}k_{2i})\frac{\partial^2 U_i}{\partial r \partial z}\right] - c_{44}\frac{1}{r}\frac{\partial^2 U_4}{\partial \theta \partial z}, \\
\sigma_{r\theta} &= \sum_{i=1}^{3}\left[(c_{11} - c_{12})\left(\frac{1}{r}\frac{\partial^2 U_i}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial U_i}{\partial \theta}\right)\right] + \frac{1}{2}(c_{11} - c_{12})\left(\frac{\partial^2 U_4}{\partial r^2} - \frac{1}{r}\frac{\partial U_4}{\partial r} - \frac{1}{r^2}\frac{\partial^2 U_4}{\partial \theta^2}\right),
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
D_r &= \sum_{i=1}^{3}\left[(e_{15} + e_{15}k_{1i} - \kappa_{11}k_{2i})\frac{\partial^2 U_i}{\partial r \partial z}\right] - e_{15}\frac{1}{r}\frac{\partial^2 U_4}{\partial \theta \partial z}, \\
D_\theta &= \sum_{i=1}^{3}\left[(e_{15} + e_{15}k_{1i} - \kappa_{11}k_{2i})\frac{1}{r}\frac{\partial^2 U_i}{\partial \theta \partial z}\right] + e_{15}\frac{\partial^2 U_4}{\partial r \partial z}, \\
D_z &= \sum_{i=1}^{3}\lambda_i (e_{15} + e_{15}k_{1i} - \kappa_{11}k_{2i})\frac{\partial^2 U_i}{\partial z^2}.
\end{aligned}
\tag{11}
$$

A ELLIPSOIDAL CAVITY UNDER REMOTE LOADING

Boundary conditions on the cavity surface

Figure 1 shows an ellipsoidal cavity in an infinite transversely isotropic piezoelectric medium under remote loading. The center of the cavity is located at the origin of the coordinate system. The ellipsoidal surface is

$$\frac{r^2}{a^2} + \frac{z^2}{b^2} = 1 \tag{12}$$

or

$$\frac{r^2}{q^2 - 1} + \frac{z^2}{q^2} = C_0^2, \tag{13}$$

where $C_0^2 = b^2 - a^2$ and $q = b / C_0$, with the unit out normal $\{n_r, 0, n_z\}$,

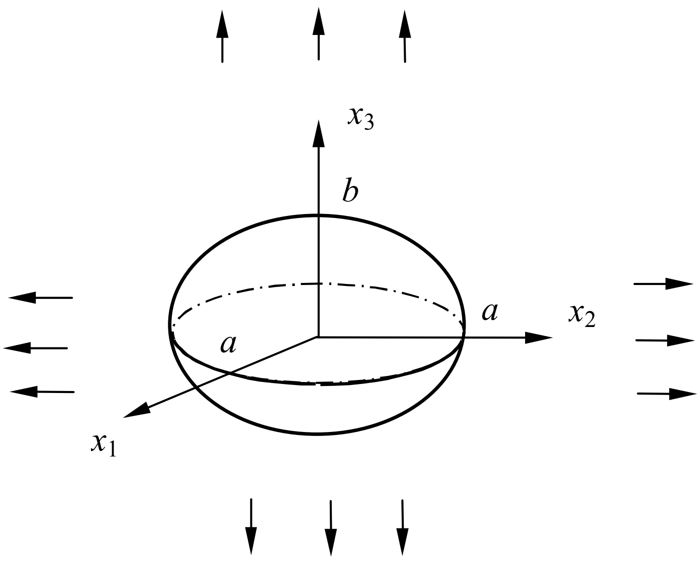

Fig. 1 Ellipsoidal cavity

$$n_r = \frac{r}{a^2 N}, \quad n_z = \frac{z}{b^2 N}, \quad N = \sqrt{\frac{r^2}{a^4} + \frac{z^2}{b^4}}. \tag{14}$$

The boundary conditions along the cavity surface

$$\begin{aligned}
&\phi = \phi^c \quad \text{(continuity of electric potential)}, \\
&\sigma_{rr} n_r + \sigma_{rz} n_z = 0,\ \ \sigma_{\theta r} n_r + \sigma_{\theta z} n_z = 0, \quad \sigma_{rz} n_r + \sigma_{zz} n_z = 0, \quad \text{(traction – free)}, \\
&D_r n_r + D_z n_z = 0 \ \ \text{(surface charge – free)},
\end{aligned} \tag{15}$$

where the superscript "*c*" denotes a quantity inside the cavity.

Applied loads

For simplicity, we consider only the axisymmetric loading in the present work. Under the remotely uniform loads of $\sigma_{rr}^{\infty}, \sigma_{zz}^{\infty}$ and electric displacement D_z^{∞}, the corresponding displacements and electric potential are

$$u_r^{\infty} = \varepsilon_{rr}^{\infty} r,\ \ u_z^{\infty} = \varepsilon_{zz}^{\infty} z, \quad \varphi = -E_z^{\infty} z + \varphi_0, \tag{16}$$

where φ_0 is a reference electric potential, and

$$\begin{aligned}
&\sigma_{rr}^{\infty} = (c_{11} + c_{12})\varepsilon_{rr}^{\infty} + c_{13}\varepsilon_{zz}^{\infty} - e_{31}E_z^{\infty}, \quad \sigma_{zz}^{\infty} = 2c_{13}\varepsilon_{rr}^{\infty} + c_{33}\varepsilon_{zz}^{\infty} - e_{33}E_z^{\infty}, \\
&D_z^{\infty} = 2e_{13}\varepsilon_{rr}^{\infty} + e_{33}\varepsilon_{zz}^{\infty} + \kappa_{33}E_z^{\infty}.
\end{aligned} \tag{17}$$

Solution

Assume that the electric field strength inside the cavity is uniform with the electric potential,

$$\varphi^c = -\hat{B}z + \varphi_0^c, \tag{18}$$

where φ_0^c is another reference electric potential and $\hat{B}$ is a constant.

In this case,

$$U_4 = 0, \tag{19}$$

and the other three harmonic potential function U_i are given by [11]

$$U_i = A_i H(r, z_i), \quad i = 1, 2, 3, \tag{20}$$

with

$$\begin{aligned}
&H(r, z_i) = \frac{1}{2}\left[z_i^2 \psi_1(q_i) + r^2 \psi_2(q_i) - C_i^2 \psi_0(q_i)\right], \\
&\psi_0(q_i) = \frac{1}{2}\ln\left(\frac{q_i + 1}{q_i - 1}\right), \quad \psi_1(q_i) = \frac{1}{2}\ln\left(\frac{q_i + 1}{q_i - 1}\right) - \frac{1}{q_i}, \quad \psi_2(q_i) = -\frac{1}{4}\ln\left(\frac{q_i + 1}{q_i - 1}\right) + \frac{1}{2}\frac{q_i}{q_i^2 - 1},
\end{aligned} \tag{21}$$

where A_i, i=1, 2, 3, are constants and the independent variables $q_i(r, z_i)$, i=1, 2, 3, are defined implicitly by

$$\frac{r^2}{q_i^2 - 1} + \frac{z_i^2}{q_i^2} = C_i^2, \quad C_i^2 = (b^2 / \lambda_i) - a^2. \tag{22}$$

Denoting $p_i^2 = b^2 / (b^2 - \lambda_i a^2)$, we have $q_i = p_i$ for points (r, z) lying on the surface of the spheroid. Substituting the above expressions into the boundary conditions, i.e., Eq. (15), yields the equations to determine the constants A_i and $\hat{B}$

$$
\begin{aligned}
&E_z^\infty - \sum_{i=1}^{3}\frac{k_{2i}}{\lambda_i}\psi_1(p_i)A_i = \hat{B} \quad \text{(continuity of electric potential)},\\
&\sum_{i=1}^{3}\left[(c_{12}-c_{11})\psi_2(p_i) - \frac{c_{44}(1+k_{1i})+e_{15}k_{2i}}{\lambda_i}\psi_1(p_i)\right]A_i = -\sigma_{rr}^\infty,\\
&2\sum_{i=1}^{3}\left[c_{44}(1+k_{1i})+e_{15}k_{2i}\right]\psi_2(p_i)A_i = \sigma_{zz}^\infty \quad \text{(traction − free)},\\
&2\sum_{i=1}^{3}\left[e_{15}(1+k_{1i})-\kappa_{11}k_{2i}\right]\psi_2(p_i)A_i = D_z^\infty - \kappa^c\hat{B} \quad \text{(surface charge − free)},
\end{aligned}
\tag{23}
$$

where κ^c is the electric permeability of the cavity. After the coefficients have been determined, the stress and the electric displacement field everywhere can be calculated from Eqs. (10) and (11).

If b approaches zero, i.e., $\alpha = b/a \to 0$, the cavity is shrunk to a penny-shaped crack. Using the kinematic and constitutive equations of Eqs. (2) and (3), the stress and the electric displacement in the crack plane can be obtained

$$
\sigma_{zz}(r,0) = \frac{2\sigma_{zz}^\infty}{\pi}\left[\frac{1}{\sqrt{(\tilde{r})^2-1}} + Arc\tan\sqrt{(\tilde{r})^2-1}\right], \quad \tilde{r} = \frac{r}{a} > a \tag{24}
$$

$$
D_z(r,0) = \frac{2(D_z^\infty - d^*)}{\pi}\left[\frac{1}{\sqrt{(\tilde{r})^2-1}} + Arc\tan\sqrt{(\tilde{r})^2-1}\right] + d^*, \tag{25}
$$

where

$$
d^* = D_z^\infty + \sigma_{zz}^\infty\tilde{M}_5 \quad \text{for a finite } \beta^* \text{ or } \beta^* \to 0 \text{ and } \alpha/\beta^* \to 0, \tag{26a}
$$

$$
d^* = \frac{D_z^\infty + \sigma_{zz}^\infty\tilde{M}_5}{1+\dfrac{\alpha}{\beta^*}} \quad \text{for } \beta^* \to 0 \text{ and } \alpha/\beta^* \text{ has a finite nonzero value }, \tag{26b}
$$

$$
d^* = 0 \quad \text{for } \beta^* \to 0 \text{ and } \alpha/\beta^* \to \infty, \tag{26c}
$$

where

$$
\beta^* = \kappa^c/\kappa_{3D}^{eff}, \quad \kappa_{3D}^{eff} = \det[\mathbf{M}^{(1)}]/\det[\mathbf{M}^{(3)}], \quad \tilde{M}_5 = \det[\mathbf{M}^{(5)}]/\det[\mathbf{M}^{(3)}], \tag{27a}
$$

$$
\begin{aligned}
&M_{1i}^{(1)} = s_i\left[c_{44}(1+k_{1i})+e_{15}k_{2i}\right], \quad M_{2i}^{(1)} = \frac{\pi i}{2}\left[c_{44}(1+k_{1i})+e_{15}k_{2i}\right], \quad M_{3i}^{(1)} = \frac{\pi i}{2}\left[e_{15}(1+k_{1i})-\kappa_{11}k_{2i}\right],\\
&M_{1i}^{(3)} = M_{1i}^{(1)}, \quad M_{2i}^{(3)} = M_{2i}^{(1)}, \quad M_{3i}^{(3)} = -s_i k_{2i},\\
&M_{1i}^{(5)} = M_{1i}^{(1)}, \quad M_{2i}^{(5)} = s_i k_{2i}, \quad M_{3i}^{(5)} = M_{3i}^{(1)}, \quad i = 1,\ 2,\ 3.
\end{aligned}
\tag{27b}
$$

Equation (27a) gives the effective dielectric constant of the material for the three-dimensional problems. The results indicate that the mechanical and electric fields are strongly dependent on the ratio of α/β^*, like the two-dimensional case in which the solution depends strongly on the ratio of α/β. The two extremes given by Eqs. (26a) and (26c) correspond, respectively, to the electrically permeable and impermeable boundary conditions along the crack faces.

Defining the Mode I intensity factors K_I^σ and K_I^D

$$K_I = \lim_{r \to a} \sqrt{2\pi(r-a)}\sigma_{zz}, \ K_D = \lim_{r \to a} \sqrt{2\pi(r-a)}D_z, \tag{28}$$

one can obtain

$$K_I = 2\sigma_{zz}^{\infty}\sqrt{\frac{a}{\pi}}, \quad K_D = 2(D_z^{\infty} - d^*)\sqrt{\frac{a}{\pi}} \ . \tag{29}$$

These results show that, as in the two-dimensional problems [14], the mode I stress intensity factor is the same as that in purely elastic media and independent of the applied electric displacement. The electric displacement intensity factor depends not only on the applied fields, but also on the material properties in terms of d^*. For the two limiting cases, we have

$$K_D = -2\sigma_{zz}^{\infty}\widetilde{M}_5\sqrt{\frac{a}{\pi}} \tag{30}$$

for electrically permeable cracks and

$$K_D = 2D_z^{\infty}\sqrt{\frac{a}{\pi}} \tag{31}$$

for electrically impermeable cracks. Using the electrically permeable or impermeable boundary conditions, Kogan et al. [11] and Huang [12] obtained the intensity factors for electrically permeable cracks, while Wang [4], Zhao et al. [7, 8] and Chen et al. [9, 10] obtained the intensity factors for electrically impermeable cracks.

ACKNOWLEDGEMENTS: This work is supported by a grant from the Research Grant Council of the Hong Kong Special Administrative Region, China. MHZ thanks HKUST for the Post-Doctoral Fellowship Matching Fund.

REFERENCES

1. Deeg, W. F. J. (1980). Ph. D. Thesis, Stanford University.
2. Sosa, H. A. and Pak, Y. E. (1990) *Int. J. Solids Struct.* **26**, 1.
3. Wang, B. (1992) *Int. J. Solids Struct.* **29**, 293.
4. Wang, B. (1992) *Int. J. Engng. Sci*. **30**, 781.
5. Dunn, M. L. (1994) *Int. J. Engng. Sci*. **32**, 119.
6. Wang, Z. K. and Huang, S. H. (1995) *Theor. Appl.Fract. Mech*. **22**, 229.
7. Zhao, M. H., Shen, Y. P., Liu, Y. J. and Liu, G. N. (1997) *Theor. Appl. Fract. Mech.* **26**, 129.
8. Zhao, M. H., Shen, Y. P., Liu, Y. J. and Liu, G. N. (1997) *Theor. Appl. Fract. Mech.* **26**, 141.
9. Chen, W. Q. and Shioya, T. (1999) *J. Mech. Phys. Solids* **47**,1459.
10. Chen, W. Q. and Shioya, T. (2000) *Int. J. Solids Struct.* **37**, 2603.
11. Kogan, L., Hui, C. Y. and Molkov V. (1996) *Int. J. Solids Struct.* **33**, 2719.
12. Huang, J. H. (1997) *Int. J. Solids Struct.* **34**, 2631.
13. Zhang, T.-Y. (1994) *Int. J. Fracture* **66**, R33.
14. Zhang, T.-Y., and Tong, P. (1996) *Int. J. Solids. Struct.* **33**, 343.
15. Zhang, T.-Y., Qian, C.-F., and Tong, P. (1998) *Int. J. Solids Struct.* **35**, 2121.
16. Wang, Z. K. and Zheng, B. L. (1995) *Int. J. Solids Struct.* **32**, 105.
17. Ding, H. J., Chen, B. and Liang J. (1996) *Int. J. Solids Struct.* **33**, 2283.