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ON FATIGUE FAILURE OF A MULTIPLE-LOAD-PATH REDUNDANT STRUCTURE

A. M. Freudenthal*

and

M. Shinozuka**

ABSTRACT

It is difficult to evaluate the survivorship function of a structure subject to fatigue consisting of a number of members from those of single specimens that can be obtained from experiment.

An attempt is therefore made to establish the upper and lower bounds of such a survivorship function under the assumption of constant amplitude fatigue and equal distribution of the load among the existing members, approximating the real process of the failure by Markovian processes.

A numerical example employing the data of fatigue tests performed on 7075 Aluminum alloy single specimens indicates that the order of magnitude of the life of a composite structure can be reasonably well predicted by the bounds established by the present method.

1. Introduction.

It is known that, in general, the observed fatigue life N of engineering materials shows such a wide scatter, both under constant and random stress amplitude (for example [1]), that from the reliability aspect it has to be treated as a random variable.

However, except for the work by Birnbaum and Saunders [2] where the statistical distributions of the life of multi-membered structures are derived under an interesting assumption concerning the deterioration of material and a paper by Heller and Heller [3], based on a linear damage accumulative rule and the assumption of exponential distribution, the investigations have so far been limited to the case of single member structures or single specimens with the emphasis on the interpretation of the stochastic process in terms of microscopic physical mechanism of

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* Professor of Civil Engineering, Columbia University, New York, N. Y.
** Associate Professor of Civil Engineering, Columbia University, New York, N. Y.

fatigue and/or phenomenological criteria of fatigue failure, although in reality most of structures in which fatigue is the critical mode of failure are multi-membered.

The purpose of this paper is to extend this line of investigation to the case of multiple-load-path structures consisting of n members (Fig. 1) of identical material and cross-sectional area A, subject to the cyclic load F of constant amplitude.

It should be pointed out that the problem is of considerable mathematical interest since the stochastic process associated with a single specimen is of Markovian type while that associated with a multiple-load-path structure is non-Markovian. This makes it extremely difficult to evaluate the survivorship function of a multiple-member structure on the basis of experimental results on single specimens. The present paper, however, shows that it is possible to find upper and lower bounds of the survivorship function of multiple-load-path structures using the result of experiments on single specimens without making assumptions such as the one used in [2] or linear damage rule employed in [3].

2. Upper and Lower Bounds.

Consider a structure consisting of n equal members combined in parallel as shown in Fig. 1. Assume for the sake of simplicity that the total load F is equally distributed among the existing members and that the failure of the structure is defined as occurring when all n members fail.

Let $P_x(N)$ denote the probability that exactly $n-x$ members still exist at the N^{th} load application. If $\lambda_x \Delta N$ denotes the probability of transition $x \rightarrow x+1$ in the interval $(N, N+\Delta N)$

$$P_x(N+\Delta N) = (1-\lambda_x \Delta N)P_x(N) + \lambda_{x-1}P_{x-1}(N)\Delta N \tag{1}$$

This is the well-known relation in the pure birth process leading to the following basic differential equations of $P_x(N)$:

$$dP_0(N)/dN = -\lambda_0 P_0(N) \tag{2}$$

$$dP_x(N)/dN = -\lambda_x P_x(N) + \lambda_{x-1}P_{x-1}(N) \qquad (x = 1, 2, \cdots, n-1) \tag{3}$$

$$dP_n(N)/dN = \lambda_{n-1}P_{n-1}(N) \tag{4}$$

with the initial conditions

$$P_0(0) = 1, \quad P_1(0) = P_2(0) = \cdots = P_n(0) = 0 \tag{5}$$

By definition, the survivorship function of the structure is

$$L(N) = 1 - P_n(N) \tag{6}$$

Unfortunately, except for n = 1, Eqs. (2) to (6) cannot be directly employed in the present problem because λ_x depends not only on x and N but also on when the previous transitions $0 \rightarrow 1$, $1 \rightarrow 2, \ldots, x-1 \rightarrow x$ have taken place (thus the process is non-Markovian). When n = 1 or when the structure consists of a single specimen, the process is Markovian since λ_0 is a function of N only and the use of Eqs. (2), (4), (5) and (6) with n = 1 produces the exact solution to the problem.

Returning to the general case, since the transition $x \rightarrow x+1$ will occur when exactly one of the $n-x$ existing members fails, the following relation is valid

$$\lambda_x = (n-x)\nu_x(1-\nu_x)^{n-x-1} \simeq (n-x)\nu_x \tag{7}$$

where ν_x is the probability of an individual member to fail at the N^{th} load application after surviving N - 1 previous applications. The approximation in Eq. (7) is based on the assumption that $\nu_x \ll 1$.

To establish the upper and lower bounds, consider the following two extreme cases: (a) the first x transitions occur at the first x applications of the load F, one at each application; and (b) the first transitions occur at the last x applications of F, one at each application.

If $\mu_0(N)$ and $\mu_x(N)$ denote the failure rates of a single specimen at the N^{th} application of F/n and F/(n-x) respectively, then one can say from physical reasoning that in case (a) ν_x is slightly less than $\mu_x(N)$ while in case (b) ν_x is slightly larger than $\mu_0(N)$. In general, ν_x is less than that in case (a) and larger than that in case (b). Hence,

$$\mu_0(N) < \nu_x(N) < \mu_x(N) \tag{8}$$

Note that $\mu_0(N), \mu_1(N), \ldots \mu_{n-1}(N)$ can be observed from experiment on single specimens subject respectively to the loads F/n, F/(n-1),..., F.

Hence a lower bound for the survivorship function of a multiple-load-path structure can be obtained in the form $[1 - P_n(N)]$ with $P_n(N)$ being the solution of Eqs. (2) to (5) replacing ν_x by $\mu_x(N)$ whereas an upper bound can be obtained as $[1 - P_n(N)]$ from the same equations replacing ν_x by $\mu_0(N)$.

It should be noted that the failure rate $\mu_x(N)$ of single specimens are related to the survivorship functions $L_x(N)$ associated with the stress level $S_x = F/\{A(N-x)\}$ in the following well-known form

$$L_x(N) = \exp\left[-\int_0^N \mu_x(\xi)d\xi\right] \qquad (x=0,1,2,\cdots,n-1) \tag{9}$$

since, as mentioned previously, if the structure consists of a single member ($n = 1$), then the process is Markovian and, similar to Eqs. (2), (4) and (6), one obtains

$$dP_0(N)/dN = -\mu_x(N)P_0(N) \tag{10}$$

$$dP_1(N)/dN = \mu_x(N)P_0(N) \tag{11}$$

$$L_x(N) = 1 - P_1(N) \tag{12}$$

which lead to Eq. (9) under the same initial conditions as in Eq. (5). Incidentally, it follows from definition and Eq. (7) that

$$\mu_0(N) = \nu_0(N) = \lambda_0(N)$$

A possible physical interpretation of the failure rate of $\mu_x(N)$ is given by Yokobori [4] from crack nucleation theory.

For the distribution function of fatigue life N at a constant stress amplitude the third asymptotic distribution of smallest values [5] is assumed. Hence, the survivorship function L(N) denoting the probability that the life is larger than N, takes the following form.

$$L(N) = \exp\left[-\left(\frac{N-N_0}{V-N_0}\right)^{\alpha}\right] \qquad (N \geqslant N_0) \tag{13}$$

where α, V and N_0 are parameters being functions of stress levels; in particular N_0 denotes the minimum number of stress applications to cause fatigue failure or the minimum fatigue life.

The application of Eq. (13) to the problem of fatigue has been developed by Freudenthal and Gumbel [6]. The result of a set of fatigue tests performed by Freudenthal, Heller and O'Leary on 7075 Aluminum alloy [7] at various stress levels is plotted in Fig. 2 on extremal probability paper where a straight line can be fitted reasonably well to the data at each stress level. This implies that Eq. (13) with the minimum fatigue life $N_0 = 0$ is in fact a reasonable assumption for the survivorship function for the material examined.

The parameter α is the slope of the straight line based on arithematical scale plotting of the ordinate and logamithmic scale plotting of the abscissa of the probability paper; it is inversely proportional to the standard deviation of $\log_{10}N$, while V denotes the characteristic fatigue life such that $L(V) = 1/e$ or the value of N at which the straight line intersects with the probability level $1/e$.

The parameters α and V are evaluated at each stress level and plotted in Figs. 3 and 4 respectively where curves are fitted to the observations by inspection for general trends. Figs. 3 and 4 indicate that α increases while V decreases with stress level.

Comparing Eq. (9) with Eq. (13) with $N_0 = 0$, one obtains

$$\mu_x(N) = \frac{\alpha_x}{V_x}\left(\frac{N}{V_x}\right)^{\alpha_x - 1} = \frac{\alpha_x}{V_x^{\alpha_x}} N^{\alpha_x - 1} \tag{14}$$

where α_x and V_x are values of α and V associated with stress levels S_x.

The following numerical example is based on the failure rate of the form of Eq. (14).

3. Numerical Example.

As a numerical example, a structure consisting of four parallel members of 7075 Aluminum alloy as shown in Fig. 1 is considered ($n = 4$). The applied load F is such that it will produce stress levels $S_1 = \pm 22.5$ ksi, $S_2 = \pm 30.0$ ksi, $S_3 = \pm 45.0$ ksi and $S_4 = \pm 90.0$ ksi. Since S_4 ($= \pm 90$ ksi) is larger than the ultimate strength (approximately 82 ksi) of the material, the failure of the entire structure will immediately follow the failure of any three of the four members. Therefore, n is equal to 3 in applying Eqs. (2) - (6).

From Figs. 3 and 4, α_x and V_x are evaluated and listed in Table 1.

For the upper bound $L^+(N)$ of $L(N)$, the use of Eqs. (2) - (6) is made with the following $\lambda_x(N)$.

$$\lambda_0(N) = 4\mu_0(N) = \lambda'_0 N^{\alpha_0 - 1} = 4\frac{\alpha_0}{V_0^{\alpha_0}} N^{\alpha_0 - 1} \tag{15a}$$

$$\lambda_1(N) = 3\mu_0(N) = \lambda'_1 N^{\alpha_0 - 1} = 3\frac{\alpha_0}{V_0^{\alpha_0}} N^{\alpha_0 - 1} \tag{15b}$$

$$\lambda_2(N) = 2\mu_0(N) = \lambda_2' N^{\alpha_0-1} = 2\frac{\alpha_0}{V_0^{\alpha_0}} N^{\alpha_0-1} \tag{15c}$$

which are obtained from Eq. (7) with ν_x replaced by $\mu_x(N)$.

The successive integrations of Eqs. (2) - (4) with n = 3 under the initial condition Eq. (5) produce

$$P_3(N) = \frac{\beta_1\beta_2}{(\beta_0-\beta_1)(\beta_0-\beta_2)}(1-e^{-\beta_0 N^{\alpha_0}}) - \frac{\beta_0\beta_2}{(\beta_0-\beta_1)(\beta_1-\beta_2)}(1-e^{-\beta_1 N^{\alpha_0}}) + \frac{\beta_0\beta_1}{(\beta_1-\beta_2)(\beta_0-\beta_2)}(1-e^{-\beta_2 N^{\alpha_0}}) \tag{16}$$

where

$$\beta_x = \lambda_x' / \alpha_0 \qquad (x = 0, 1, 2) \tag{17}$$

From Table 1 and Eq. (14), $\alpha_0' = 2.00$, $\lambda_0' = 2.54 \times 10^{-15}$, $\lambda_1' = 3/4 \times \lambda_0' = 1.91 \times 10^{-15}$ and $\lambda_2' = 1/2\,\lambda_0' = 1.27 \times 10^{-15}$.

Using these values in Eqs. (15), (16) and (17) the upper bound $L^+(N) = 1 - P_3(N)$ is found to be

$$L^+(N) = 3e^{-1.27\times10^{-15}N^2} - 8e^{-9.52\times10^{-16}N^2} + 6e^{-6.35\times10^{-16}N^2} \tag{18}$$

For the lower bound $L^-(N)$, use can be made of Eqs. (2) - (6) with

$$\lambda_0 = 4\mu_0(N) = \lambda_0' N^{\alpha_0-1} \tag{19a}$$

$$\lambda_1 = 3\mu_1(N) = \lambda_1' N^{\alpha_1-1} \tag{19b}$$

$$\lambda_2 = 2\mu_2(N) = \lambda_2' N^{\alpha_2-1} \tag{19c}$$

where λ_x are obtained from Eq. (7) with ν_x replaced by $\mu_x(N)$ and

$$\lambda_x' = (4-x)\,\alpha_x / V_x^{\alpha_x} \qquad (x = 0, 1, 2) \tag{20}$$

Although it is feasible to perform integrations of Eqs. (2) - (4) successively with λ_x of the form of Eq. (19) and hence evaluate $L^-(N)$ numerically with the aid of a computer, possible propagation of errors involved as one proceeds from Eq. (2) to Eq. (4) makes the result unreliable.

Hence, in the present investigation, quantities $\mu_x^*(N)$ ($x = 0,1,\dots n-1$) are introduced to satisfy the following relations at least for N less than or equal to some appropriate value N' beyond which the survivorship function of the structure is expected to be so small that it is of little practical use:

$$\mu_x^*(N) = \alpha_0 N^{\alpha_0-1}/V_x^{*\alpha_0} \geq \mu_x(N) = \alpha_x N^{\alpha_x-1}/V_x^{\alpha_x} \qquad (x = 0, 1, \dots, n-1) \quad (N \leq N') \tag{21}$$

where V_x^* will be defined later. If $\mu_x^*(N)$ are used in place of $\mu_x(N)$ in applying Eqs. (2) - (7), the resulting survivorship function is evidently a lower bound within the domain of N where Eq. (21) is valid ($N \leq N'$).

Note that the power of N in the expression of $\mu_x^*(N)$ is equal to $\alpha_0 - 1$ for all x and this makes it possible to evaluate a lower bound $L^-(N)$ in a closed form.

To find $\mu_x^*(N)$ satisfying Eq. (21), consider fictious survivorship functions

$$L_x^*(N) = \exp\left[-\left(\frac{N}{V_x^*}\right)^{\alpha_0}\right] \qquad (x = 0, 1, 2, \dots, n-1) \tag{22}$$

where V_x^* are chosen in such a way that

$$\mu_x^*(N') = \mu_x(N') \tag{23}$$

or

$$V_x^{*\alpha_0} = \frac{\alpha_0}{\alpha_x} N'^{\,\alpha_0-\alpha_x} V_x^{\alpha_x} \tag{24}$$

Then, it can be shown that, the failure rates $\mu_x^*(N)$ associated with $L_x^*(N)$ are larger than or equal to $\mu_x(N)$ associated with $L_x(N)$ as long as $N \leq N'$ since

$$\mu_x^*(N) = \alpha_0 N^{\alpha_0-1}/V_x^{*\alpha_0} = \alpha_x \left(\frac{N}{N'}\right)^{\alpha_0-\alpha_x} \left(\frac{N^{\alpha_x-1}}{V_x^{\alpha_x}}\right) = \left(\frac{N}{N'}\right)^{\alpha_0-\alpha_x} \mu_x(N) \tag{25}$$

In particular,

$$L_o^*(N) = L_o(N), \quad \mu_o^*(N) = \mu_o(N), \quad \lambda^* = \lambda_o' \tag{26}$$

For the present numerical example, N' is chosen to be 5 x 10^7. This choice seems appropriate since the lower bound at N = 5 x 10^7 turns out very small as shown in the following. The values of V_x associated with N' = 5 x 10^7 are computed from Eq. (24) and listed in Table 1.

Hence, for the lower bound $L^-(N)$, use is made of Eqs. (2) - (6) with

$$\lambda_o(N) = 4\mu_o^*(N) = \lambda_o^* N^{\alpha_o - 1} = 4\frac{\alpha_o}{V_o^{*\alpha_o}} N^{\alpha_o - 1} \tag{27a}$$

$$\lambda_1(N) = 3\mu_1^*(N) = \lambda_1^* N^{\alpha_o - 1} = 3\frac{\alpha_o}{V_1^{*\alpha_o}} N^{\alpha_o - 1} \tag{27b}$$

$$\lambda_2(N) = 2\mu_2^*(N) = \lambda_2^* N^{\alpha_o - 1} = 2\frac{\alpha_o}{V_2^{*\alpha_o}} N^{\alpha_o - 1} \tag{27c}$$

where, from Table 1 and Eq. (25), $\alpha_o = 2.00$, $\lambda_o^* = 2.54 \times 10^{-15}$, $\lambda_1^* = 4.58 \times 10^{-13}$ and $\lambda_2^* = 1.07 \times 10^{-5}$.

The successive integrations produce $P_3(N)$ of the same form as Eq. (16) with

$$\beta_x = \lambda_x^* / \alpha_o \tag{28}$$

Using the values of α_o and λ_i^* mentioned above, one can obtain the lower bound $L^-(N)$

$$L^-(N) = 1.00555\, e^{-1.27\times10^{-15}N^2} - .00555\, e^{-2.29\times10^{-13}N^2} + 1.01\times10^{-17} e^{-5.35\times10^{-6}N^2} \tag{29}$$

It is noted that, for $\lambda_2(N)$ to be less than unity, $N < 9.34 \times 10^4$. However, the term that essentially involves $\lambda_2(N)$, that is, the last term of the right hand side of Eq. (29) is negligible compared with the first two. Hence, the lower bound given in Eq. (29) is considered valid for $N < N' = 5 \times 10^7$.

The upper and lower bounds given respectively in Eqs. (18) and (29)

are plotted in Fig. 5 indicating that the bounds predict the life of the structure at least within an order of magnitude.

4. Discussion.

(1) Consider the probability function $L^{(1)}(N)$ of the number of stress cycles to first failure among initially existing members. Then, it can be shown that

$$L^{(1)}(N) = e^{-1.27\times10^{-15}N^2} \tag{30}$$

is a lower bound of the survivorship function L(N) of the entire structure since failure of the first member does not always produce the total failure of the structure. Comparison of Eq. (30) with Eq. (29) indicates that $L^-(N)$ and $L^{(1)}(N)$ are practically identical for the domain of N in question. This is due to the fact that in the present example the structure considered consists only of four members and the load redistribution after the failure of first member drastically increases the stress level in the remaining members so that the successive failures of the member leading to the total failure are likely to follow. This however does not seem to be the case when the structure consists of a large number of members.

(2) In deriving Eqs. (2) - (4), an assumption is made tacitly that each load application produces failure of no more than one among the existing members, although in reality a single load application can possibly cause failure to more than one member if the failure of members is instantaneous so that the successive failures of one member after another can take place in a finite duration of load application. The error involved in this assumption seems negligible since, if the (k + 1)-th failure is likely to follow the k-th failure during the application of the N-th load, meaning that p_{k+1} is very close to unity, then the assumption forces the (k + 1)-th failure to occur at the (N + 1)-th load application of the load hence introducing possible error of unity in counting the life of the structure $N \gg 1$. As pointed out by Cornell [8], however, when the applied load is also a random variable, this argument is not quite valid for the following reason. At the early stage of life, when all the members still exist and the fatigue damage is not yet appreciable, an unusually large load (with small frequency of occurrence) is required to cause failure to one of the members. Once it occurs, however, it will most probably cause successive failures of the members in a chain reaction, hence destroying the entire structure. Under the assumption, however, this load is removed after producing failure to only one member and the following load is most likely to be much smaller than the preceding one. Hence the chain reaction of failures may not occur and the error involved in such an assumption may be not only significant but also unconservative.

(3) The fact that α increases with stress level, combined with the

assumption $N_o = 0$, gives rise to the following difficulty: Considering two straight lines in Fig. 2 with $\alpha = \alpha_1$ and $\alpha_2 (> \alpha_1)$ corresponding to stress levels S_1 and $S_2 (> S_1)$ respectively, these lines must intersect at some value of N, say N^*, since $\alpha_1 < \alpha_2$. Then, for $N < N^*$, the specimen subject to S_1 has a smaller probability of survival than the specimen subject to $S_2 (> S_1)$, which is physically impossible.

The difficulty can be removed by introducing the minimum life $N_o (> o)$ as in [1]. In the present paper, however, N_o is assumed to be zero since this assumption not only simplifies the analysis but also usually produces a conservative result.

5. Conclusion.

A method is presented for the estimation of the upper and lower bounds of the survivorship function of a multi-member structure subject to a cyclic load of constant amplitude under the assumption of equal load distribution. A numerical example indicates that the present method seems to predict the life of the composite structure reasonably well at least within an order of magnitude.

6. Acknowledgment.

This work was done under the auspices of the Institute for the Study of Fatigue and Reliability sponsored jointly by the Office of Naval Research, Air Force Materials Laboratory and Advanced Research Project Agency under Contract Nonr 266(91).

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TABLE I

Values of Parameters α_x, V_x and V_x^*

	Stress Level	α_x	V_x	V_x^*
0	22.5 ksi	2.00	5.62×10^7	$5.62 \times 10^7 = V_o$
1	30.0 ksi	2.70	7.93×10^6	3.62×10^6
2	45.0 ksi	4.30	3.10×10^5	6.12×10^2

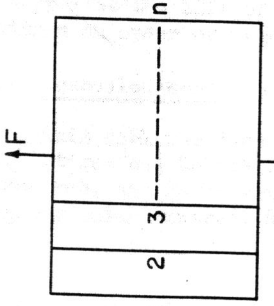

Fig. 1 A Multiple-Load-Path Redundant Structure

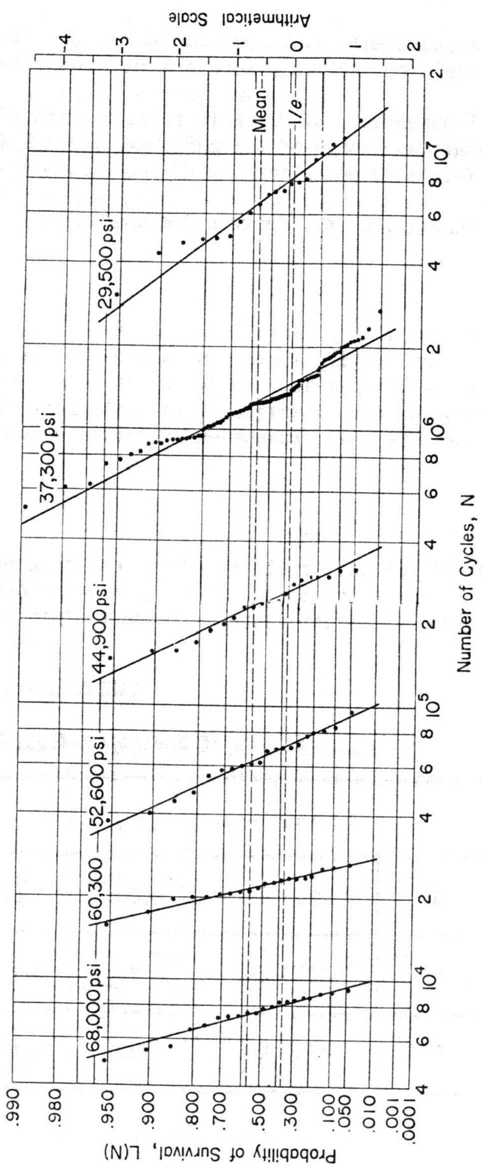

Fig. 2 Survivorship Functions of 7075 Aluminim Alloy

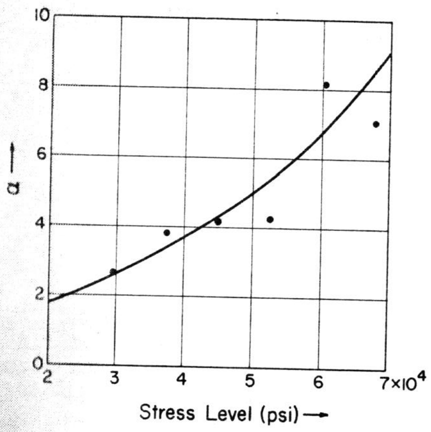

Fig. 3 Parameter α as a Function of Stress Level

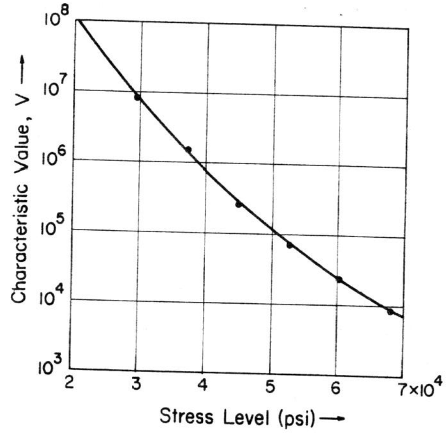

Fig. 4 Parameter V as a function of Stress Level

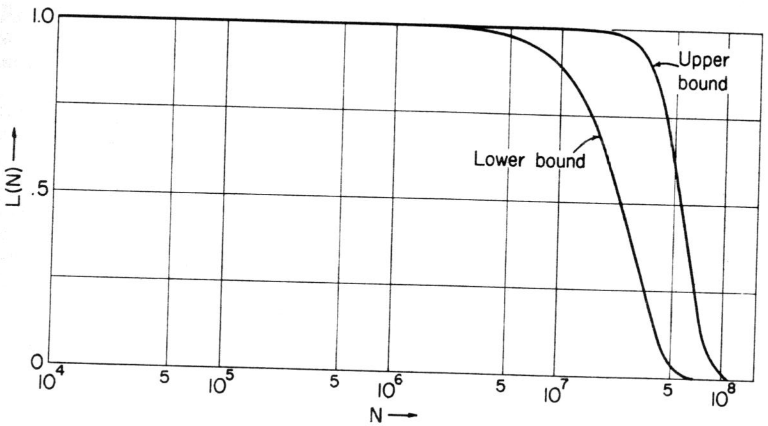

Fig. 5 Upper and Lower Bounds of Survivorship Function