DII-4

FRACTURE UNDER BIAXIAL CONDITIONS IN THE PRESENCE OF A CRACK

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ABSTRACT

The principle of balance of energies at the onset of rapid crack propagation is used to determine the fracture criterion under uniaxial fracture conditions. The determination of the term for the dissipation of plastic energy involves using properties of the stress-strain curve. The Von Mises condition namely, that the octahedral shear strain remains the same under both uniaxial and biaxial loading enables one to determine the biaxial stress-strain curve from the uniaxial stress-strain curve and the parameter expressing the degree of biaxiality. The uniaxial Fracture curve as determined by this method gives a good correlation with test data. The predicted biaxial fracture curve qualitatively behaves essentially along the lines of observation in the normal engineering range of stresses. That is, the material behaves as if it were more crack sensitive in biaxial loading.

INTRODUCTION

An important engineering problem in space technology is that of fracture of pressure vessels which are simultaneously subjected to longitudinal and circumferential stresses. Howsoever carefully the material is selected and the pressure vessel fabricated, instances are known where a crack has gone undetected and lead to a subsequent catastrophic failure. Till recently, much of the attention in fracture has been focussed on the uniaxial problem; the biaxial problem receiving scant little attention due to experimental difficulties and theoretical complexities. In the uniaxial fracture the energy method saw considerable development in the hands of Irwin (Ref. 1) and his associates and their work forms the corner stone of development in macroscopic fracture in the United States. In its original form their method did not receive much experimental support except in the case of very high strength materials. It appears that

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the reason for this is that they have not taken into consideration the variations in the plastic energy density but assumed that it could be inferred from the strain energy release rate. It is proposed in what follows to make an estimate of the rate of plastic energy dissipation at the onset of rapid crack growth and use this term in the incremental energy balance equation. This equation for metallic materials which manifest some ductility prior to fracture may be stated thus

$$\frac{\delta U_e}{\delta \ell} \geqslant \frac{\delta U_p}{\delta \ell} \tag{1.1}$$

where the left hand side term represents the incremental changes in the elastic energy release rate and the right hand side the incremental energy dissipation in plastic deformation. It is assumed, in line with current practice, that the surface energy term is small compared with the plastic energy term and hence can be neglected.

The density of plastic energy dissipated may be estimated in the following manner. The stress-strain relation is represented for many engineering materials to a good approximation by the equation

$$\varepsilon = \frac{\sigma}{E} + C\cdot\left(\frac{\sigma}{\sigma_s}\right)^n \tag{1.2}$$

where ε and σ are the strain and stress respectively. n is the strain hardening coefficient and C and σ_s are related constants. In particular it will be noted that since $\varepsilon - \frac{\sigma}{E}$ is the plastic strain if σ_s is chosen as the stress at yield point C will have the value 0.2%. The energy dissipated in plastic deformation in a unit volume is

$$\int_0^{\sigma_p} \sigma\, d\varepsilon - \frac{\sigma_p^2}{2E} = C\cdot\frac{n}{n+1}\,\sigma_s\left(\frac{\sigma_p}{\sigma_s}\right)^{n+1} \tag{1.3}$$

where σ_p is the stress in the plastic range of the stress-strain curve. Let us now consider a thin sheet of unit thickness with width 2W having a central crack of length 2ℓ. If such a specimen is subjected to a stress σ_∞ far away from the crack in a direction perpendicular to the crack, the elastic solution indicated by Irwin (Ref.1) is applicable. Defining the zone of plastic deformation as the region in which the maximum principle stress σ_1 exceeded the yield point stress σ_{yp} it can be shown that the polar radius of the contour enclosing the region is given approximately by

$$r = \frac{K^2}{2\sigma_{yp}^2}\cdot\cos^2\frac{\theta}{2}\left\{1+\sin\frac{\theta}{2}\right\}^2 \tag{1.4}$$

and that

$$\sigma_1 = \frac{K}{\sqrt{2r}}\cdot\cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\right) \tag{1.5}$$

where

$$K^2 = \sigma_\infty^2\cdot\frac{2W}{\pi}\tan\frac{\pi\ell}{2W} = \sigma_\infty^2\frac{2W}{\pi}\tan\frac{\pi\lambda}{2} \tag{1.6}$$

Equation (1.6) is a useful finite width approximation proposed by Irwin (Ref.1) following Westergaard's analysis of a series of colinear cracks. The above equations, although held valid for sharp cracks are assumed to be good approximations for a crack with finite crack tip radius ρ which is very small compared to crack length. One significant aspect of crack growth may now be pointed out. Before the onset of rapid crack growth, howsoever small, there is in general some slow crack growth. Now consider an element immediately ahead of the 60° line along which the elastic stress σ_1 is a maximum. As the applied stress for away from the crack gradually increases from 0 value, the stress in this element also increases and at some stage goes into the plastic region. Since it is very close to the 60° line, the stress there will be very close to the plastic stress corresponding to the elastic stress at the same radius on the 60° line. If at some stage there is a slow crack growth $\delta\ell$ in this process, the element under consideration may be expected to go from just ahead of the line to just behind the line where, by definition, the stress is less than the plastic stress corresponding to σ_1. Since the stress in the element is lower now, it is unloading and the unloading occurs along the elastic line. Therefore in discussing the stability condition, and evaluating the incremental change in the energy of plastic deformation, one need only take into account the plastic energy density changes in a strip of length $\delta\ell$ along the radial line with $\theta = 60°$.

Consider the element shown in figure 1. The total change in the incremental plastic energy of dissipation is

$$\delta U_p = 4\int_0^{r_1} C\cdot\frac{n}{n+1}\cdot\sigma_{yp}\left(\frac{\sigma_p}{\sigma_{yp}}\right)^{n+1} dr\,\sin\theta\,\delta\ell \tag{1.7}$$

where

$$r_1 = \frac{K^2}{2\sigma_{yp}^2}\cos^2\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\right)^2$$

$$= 0.54\left(\frac{\sigma_n}{\sigma_{yp}}\right)^2(1-\lambda)^2 w\cdot\tan\frac{\pi\lambda}{2} \tag{1.8}$$

where σ_n is the nominal net section stress. The elastic energy release rate for this case is given by Irwin (Ref.1) as

$$\delta U_e = \frac{2\sigma_\infty^2}{E}\,2W\cdot\tan\frac{\pi\lambda}{2}\cdot\delta\ell \tag{1.9}$$

It is therefore clear that the condition for the onset of rapid crack propagation is

$$4C\cdot\frac{n}{n+1}\cdot\sigma_{yp}\cdot\sin\theta\int_0^{r_1}\left(\frac{\sigma_p}{\sigma_{yp}}\right)^{n+1} dr = \frac{2\sigma_\infty^2}{E}\cdot 2w\tan\frac{\pi\lambda}{2}$$

or

$$\int_0^{r_1}\left(\frac{\sigma_p}{\sigma_{yp}}\right)^{n+1} dr = \frac{(n+1)}{nC}\cdot\frac{w}{E\,\sigma_{yp}}\cdot\frac{1}{\sin\theta}\,\sigma_n^2(1-\lambda)^2\tan\frac{\pi\lambda}{2} \tag{1.10}$$

σ_p the actual stress in the plastic enclave at each point is dependent on σ_n. No precise expressions are available which will relate σ_p to σ_n. However an equation of the type proposed by Hardrath & Ohman (Ref.2) and modified to better fit the boundary conditions (Ref.3) may be used as a good approximation. This gives

$$\sigma_p = \sigma_n \left\{ 1 + (K_{T_n} - 1)\frac{E_T}{E} \right\} \qquad (1.11)$$

where K_{T_n} is the theoretical net section stress concentration factor at the element and E_T is the tangent modulus at the element where the stress is σ_p. It can be seen that for a sharp crack

$$K_{Tn} = \frac{\sigma_1}{\sigma_n} = \frac{1.47}{2}\left(\frac{w}{r}\right)^{\frac{1}{2}} \tan\frac{\pi\lambda}{2} \qquad (1.12)$$

where r is the polar radius of the element from the crack tip. For a crack with finite tip radius, Isida and Itagaki (Ref.4) give

$$K_{Tn} - 1 \approx (1-\lambda)\left[2\left(\frac{\ell}{\rho}\right)^{\frac{1}{2}}(1 + 0.5948\lambda^2) + 0(\lambda^4)\right] \qquad (1.13)$$

From the equation for stress-strain curve it can be seen that

$$\frac{E}{E_T} = 1 + c\cdot n \cdot \frac{E}{\sigma_{yp}}\left(\frac{\sigma_p}{\sigma_{yp}}\right)^{n-1} \qquad (1.14)$$

Substituting equations (1.11), (1.12) and (1.13) in (1.10), we obtain

$$\int_0^{r_1}\left(\frac{\sigma_n}{\sigma_{yp}}\right)^{n+1}\left[1 + \frac{(K_{Tn}-1)}{1 + c\cdot n\frac{E}{\sigma_{yp}}\left(\frac{\sigma_p}{\sigma_y}\right)^{n-1}}\right]^{n+1} \cdot dr$$

$$= \frac{2}{\sqrt{3}} \cdot \frac{(n+1)}{nc} \cdot \frac{w}{E\,\sigma_{yp}} \cdot \sigma_n^2 (1-\lambda)^2 \cdot \tan\frac{\pi\lambda}{2} \qquad (1.15)$$

This is the basic equation whose solution will yield a pair σ_n and λ, when the other parameters are given. Because of the implicit nature of the integral only a graphical solution is possible. This will be discussed elsewhere and for the present we shall use only an approximation to equation (1.10). Equation (1.10) is the basic equation proposed here in the sense that equations (1.11), (1.12) and (1.13) may be different in different cases but equation (1.10) remains the same. In a typical case, by graphical methods one finds that the area represented by the integral in equation (1.10) can be estimated to a fair approximation by the term

$$\frac{1}{10}\left[\frac{1}{2}\left\{\left(\frac{\sigma_{po}}{\sigma_{yp}}\right)^{n+1} + 1\right\}\right]\left\{0.54\left(\frac{\sigma_n}{\sigma_{yp}}\right)^2(1-\lambda)^2 w \tan\frac{\pi\lambda}{2}\right\}$$

the term within the square brackets giving a straight line approximation between σ_{po} and σ_{yp} at the crack tip and at the outer radius r_1 respectively. σ_{po} is the plastic stress at the crack tip. Using this approximate value of the integral we obtain

$$\frac{1}{20}\left\{\left(\frac{\sigma_{po}}{\sigma_{yp}}\right)^{n+1} + 1\right\} \times 0.54\left(\frac{\sigma_p}{\sigma_{yp}}\right)^2(1-\lambda)^2 w \tan\frac{\pi\lambda}{2} = \frac{n+1}{nc}\cdot\frac{w}{E\,\sigma_{yp}}\cdot\frac{1}{\sin\theta}\,\sigma_n^2(1-\lambda)^2\tan\frac{\pi\lambda}{2}$$

Therefore, $$\left(\frac{\sigma_{po}}{\sigma_{yp}}\right)^{n+1} = \left\{42.7\cdot\frac{(n+1)}{nc}\cdot\frac{\sigma_{yp}}{E}\right\} - 1 \qquad (1.16)$$

Since all the terms on the right hand side are known, σ_{po} the approximate actual stress at the crack tip is known. We next have to relate σ_{po}, the stress at the crack tip to the nominal net section stress. This is done through equation (1.11), (1.12) or (1.13) and (1.14). Since most cracks have a finite crack tip radius no matter how small, equation (1.13) seems to be the appropriate relation to use for theoretical net section stress concentration factor. However, the crack tip radius is not known with any degree of precision. It is strongly influenced by the plastic deformation. Since the longer the crack length, the smaller the stress needed to attain the onset of rapid crack propagation, it appears that the actual crack tip radius should be proportional to some inverse function of crack length. The simplest assumption one can make is

$$\rho = \rho_0\,\lambda^{-1}$$

Thus for our purposes we have

$$K_{Tn} - 1 \approx (1-\lambda)\left\{\frac{2}{\rho_0^{\frac{1}{2}}}\cdot\frac{\ell^{\frac{1}{2}}}{\lambda^{-\frac{1}{2}}}(1 + 0.595\lambda^2)\right\}$$

$$\approx A\lambda(1-\lambda)(1 + 0.595\lambda^2) \qquad (1.17)$$

The constant A will be treated as an empirical constant to be determined from test data. We therefore obtain by substitution

$$\sigma_n = \frac{\sigma_{po}}{1 + \dfrac{A\lambda(1-\lambda)(1+0.595\lambda^2)}{1 + cn\cdot\frac{E}{\sigma_{yp}}\left(\frac{\sigma_p}{\sigma_{yp}}\right)^{n-1}}} \qquad (1.18)$$

Since all the terms on the right hand side are known except the constant A, it can be determined from a test point. In order to illustrate the form of the curve predicted by the equation, an example will be worked out for fracture strength of 2024 T3 alloy sheet specimen for which data is given in NACA 3816 (Ref.5). The equation for the stress strain curve is found to be

$$\varepsilon = \frac{\sigma}{E} + 2\times10^{-3}\left(\frac{\sigma}{\sigma_{yp}}\right)^{9.53}$$

where σ_{yp} = 50,000 psi

From equation (1.16)

$$\left(\frac{\sigma_{po}}{\sigma_{yp}}\right)^{n+1} = 42.7 \times \frac{n+1}{nc}\cdot\frac{\sigma_{yp}}{E} - 1 = 110$$

$$\therefore \quad \frac{\sigma_{po}}{\sigma_{yp}} = 1.56$$

$$\therefore \quad \sigma_p = 78000\,\text{PSi}$$

Therefore equation (1.18) gives

$$\sigma_n = \frac{78000}{1+B\lambda(1-\lambda)(1+0.595\lambda^2)}$$

Expressing σ_n non-dimensionally by dividing it by the nominal ultimate uniaxial tensile strength of $\sigma_{ul} = 73{,}000$, we obtain

$$\frac{\sigma_n}{\sigma_{ul}} = \frac{1.08}{1+B\lambda(1-\lambda)(1+0.595\lambda^2)}$$

In particular for $\lambda = 0.4$ $\frac{\sigma_n}{\sigma_{ul}} = 0.62$ from test data as shown in Fig.2.

Hence B=2.82. With this value of B, the predicted curve and the corresponding test data are shown in Fig.2. It will be noted that in the test range, the curve behaves substantially as a straight line with a slope of a=1/4.73. The curve also defines a crack length of λ_o=0.03 below which the crack has no influence on the net section strength of the material in the sense that the full ultimate tensile strength is developed on the net section. Defining an effective fracture toughness index as $\sigma_n\lambda^{1/a}$, we have for this material with this thickness, a fracture toughness index of 73,000 x 0.62 $(0.4)^{1/4.73}$ = 37,400. psi. The magnitude of the index is a measure of the fracture toughness. The higher the value, the tougher the material from the standpoint of strength in the presence of cracks. For extremely ductile materials, "a" tends to infinity and the fracture toughness index tends to the ultimate tensile strength at that temperature.

SECTION 2

The relationship between uniaxial and biaxial stress-strain curves

The apparent stress-strain curve for a material under biaxial stresses, (ie) the relationship between the longitudinal stress and the longitudinal strain in the presence of a transverse stress has in general a somewhat steeper slope in the elastic range. For the same value of plastic strain, the stress in the plastic range will be also higher under biaxial conditions as compared to uniaxial conditions. If we assume that the Von Mises criterion is valid, that is, the relationship between octahedral shear stress and plastic octahedral shear strain is independent of the stress-state we can derive a transformation condition to obtain the biaxial stress-strain curve from uniaxial stress-strain curve. This means that for every point on the uniaxial stress-strain curve, a point on the biaxial stress-strain curve can be found that has the same value of the octahedral shear stress and plastic octahedral shear strain. Since the material parameters in the form of the stress-strain curve are a basic part of the theory of fracture proposed here, it is evident that a knowledge of the uniaxial fracture enables us to predict the biaxial fracture. This section essentially follows the contents of the Douglas Report SM-38958 (Ref.6).

Consider a cubic element subjected to the stress system σ_1 and σ_2 and let the stress state parameter be defined as

$$\alpha = \frac{\sigma_2}{\sigma_1} \qquad (2.1)$$

To convert the uniaxial stress-strain curve to the biaxial curve, it will be necessary to consider the elastic and plastic strains separately as shown in figure 3. The equation for the elastic strain is

$$\varepsilon_{1e} = \frac{1}{E}\left\{\sigma_1 - \mu\,\sigma_2\right\} = \frac{\sigma_1}{E}(1-\mu\alpha) \qquad (2.2)$$

where μ is the Poisson's ratio.

Denoting by subscripts U and B, uniaxial and biaxial conditions respectively, we have from the Von Mises condition

$$(\tau_{oct})_B = (\tau_{oct})_U$$

$$(\gamma_{Poct})_B = (\gamma_{Poct})_U \qquad (2.3)$$

The octahedral shear stress is given by

$$(\tau_{oct}) = \frac{1}{3}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}$$

and since σ_3 is assumed to be zero here, we have

$$(\tau_{oct}) = \frac{1}{3}\sqrt{\sigma_1^2(1-\alpha)^2+\sigma_1^2\alpha^2+\sigma_1^2}$$

$$= \frac{\sigma_1}{3}\sqrt{2}\,(1-\alpha+\alpha^2)^{\frac{1}{2}} \qquad (2.4)$$

For uniaxial stressing $\alpha = 0$ Therefore

$$(\tau_{oct})_U = \frac{\sqrt{2}}{3}\sigma_U \qquad (2.5)$$

and since it is assumed that the uniaxial and biaxial octahedral stresses are the same, it follows from (2.4) and (2.5) that

$$\sigma_1 = \sigma_U(1-\alpha+\alpha^2)^{-\frac{1}{2}} \qquad (2.6)$$

Thus equation (2.6) relates the biaxial longitudinal stress σ_1 to the uniaxial stress σ_U and the stress state parameter α.

We now make an additional assumption that the plastic strains produce no change in volume. Thus

$$\Delta V = \varepsilon_{1P} + \varepsilon_{2P} + \varepsilon_{3P} = 0 \qquad (2.7)$$

The octahedral plastic, shear strain is

$$\gamma_{Poct} = \frac{2}{3}\left\{(\varepsilon_{1P}-\varepsilon_{2P})^2+(\varepsilon_{2P}-\varepsilon_{3P})^2+(\varepsilon_{3P}-\varepsilon_{1P})^2\right\}^{\frac{1}{2}} \qquad (2.8)$$

Using relation (2.7), equation (2.8) becomes

$$(\gamma_{Poct})_B = \frac{2}{\sqrt{3}}\left(\varepsilon_{1P}^2+\varepsilon_{2P}^2+\varepsilon_{3P}^2\right)^{\frac{1}{2}} \qquad (2.9)$$

Define $\frac{\varepsilon_{2P}}{\varepsilon_{1P}} = \beta_1$ and $\frac{\varepsilon_{3P}}{\varepsilon_{1P}} = \beta_2$ to obtain

$$(\gamma_{Poct})_B = \frac{2}{\sqrt{3}}\varepsilon_{1P}(1+\beta_1^2+\beta_2^2)^{\frac{1}{2}} \qquad (2.10)$$

We now have to determine β_1 and β_2 in terms of the stress state parameter α. The deformation theory of plasticity which is consistent with equation (2.7) gives

$$\begin{aligned}\varepsilon_{1P} &= A_1\left(\sigma_1 - \frac{\sigma_2}{2}\right) = \frac{A_1}{2}\sigma_1(2-\alpha)\\ \varepsilon_{2P} &= A_1\left(-\frac{\sigma_1}{2} + \sigma_2\right) = \frac{A_1}{2}\sigma_1(-1+2\alpha)\\ \varepsilon_{3P} &= A_1\left(-\frac{\sigma_1}{2} - \frac{\sigma_2}{2}\right) = \frac{A_1}{2}\sigma_1(-1-\alpha)\end{aligned} \qquad (2.11)$$

Thus we obtain

$$\beta_1 = \frac{\varepsilon_{2P}}{\varepsilon_{1P}} = \frac{-1+2\alpha}{2-\alpha} \qquad (2.12)$$

and

$$\beta_2 = \frac{\varepsilon_{3P}}{\varepsilon_{1P}} = \frac{-1-\alpha}{2-\alpha}$$

Substituting (2.12) in (2.10) gives

$$(\gamma_{Poct})_B = \frac{2\sqrt{2}}{(2-\alpha)}\varepsilon_{1P}(1-\alpha+\alpha^2)^{\frac{1}{2}} \qquad (2.13)$$

For uniaxial stressing $\alpha = 0$ and equation (2.13) gives

$$(\gamma_{Poct})_U = \sqrt{2}\,\varepsilon_{1P} \qquad (2.14)$$

Since it is assumed that the octahedral shear strains are the same it follows from (2.13) and (2.14) that

$$(\varepsilon_{1P})_B = (\varepsilon_{1P})_U\left(\frac{2-\alpha}{2}\right)(1-\alpha+\alpha^2)^{-\frac{1}{2}} \qquad (2.15)$$

Equation (2.15) solves for the biaxial strain in the longitudinal direction in terms of the uniaxial longitudinal strain and the stress state parameter α. The total strain can be obtained by adding the biaxial elastic strain to biaxial plastic strain. Thus

$$\varepsilon_{1B} = \frac{\sigma_1}{E}(1-\mu\alpha) + \frac{1}{2}(2-\alpha)(1-\alpha+\alpha^2)^{-\frac{1}{2}}(\varepsilon_{1P})_U \qquad (2.16)$$

The uniaxial stress-strain curve can be expressed in the form

$$\varepsilon_U = \frac{\sigma_U}{E} + c\left(\frac{\sigma_U}{\sigma_S}\right)^n \qquad (2.17)$$

where $c\left(\frac{\sigma_U}{\sigma_S}\right)^n$ is the plastic strain $(\varepsilon_U)_P$.

Subsituting this form in equation (2.16), we obtain

$$\varepsilon_{1B} = \frac{\sigma_1}{E}(1-\mu\alpha) + \frac{1}{2}(2-\alpha)(1-\alpha+\alpha^2)^{-\frac{1}{2}}c\cdot\left(\frac{\sigma_U}{\sigma_S}\right)^n \qquad (2.18)$$

Substituting for σ_U in terms of a σ_1 from equation (2.6), we obtain

$$\begin{aligned}\varepsilon_{1B} &= \frac{\sigma_1}{E}(1-\mu\alpha) + \frac{1}{2}(2-\alpha)(1-\alpha+\alpha^2)^{-\frac{1}{2}}c\left\{\frac{\sigma_1}{\sigma_S}(1-\alpha+\alpha^2)^{\frac{1}{2}}\right\}^n\\ &= \frac{\sigma_1}{E/(1-\mu\alpha)} + c\left(1-\frac{\alpha}{2}\right)(1-\alpha+\alpha^2)^{\frac{n-1}{2}}\left(\frac{\sigma_1}{\sigma_S}\right)^n\end{aligned} \qquad (2.19)$$

Let $E/(1-\mu\alpha) = E_B$ and $c\left(1-\frac{\alpha}{2}\right)(1-\alpha+\alpha^2)^{\frac{n-1}{2}} = C_B$

The equation for the apparent stress-strain curve under biaxial conditions is therefore

$$\varepsilon_{1B} = \frac{\sigma_1}{E_B} + C_B\left(\frac{\sigma_1}{\sigma_S}\right)^n \qquad (2.20)$$

In otherwords, since $E_B > E$, within the elastic limit, the material behaves as if it were stiffer. The strain hardening coefficient "n" remains the same. Figure 4 shows the comparison between the theory and test data for a thin walled cylinder with $\alpha = 0.4236$. Figure 5 shows the comparison between the theory and test data for a thin walled sphere. The comparison between the theory and the tests may be considered fair.

SECTION 3

Biaxial Fracture Relation

We shall now obtain an expression for biaxial fracture by using the relationships between the uniaxial and the corresponding biaxial stress-strain curve. The basic equation for fracture in the present theory is,

$$\sigma_n = \frac{\sigma_P}{1 + \dfrac{A\lambda(1-\lambda)(1+0.595\lambda^2)}{1 + CN\dfrac{E}{\sigma_{yp}}\left(\dfrac{\sigma_P}{\sigma_{yp}}\right)^{n-1}}}$$

It will be recalled that the factor A is dependent upon the specimen width and the crack tip radius. The effect of a stress in the perpendicular direction is to make the aspect ratio of the crack finer (ie) tend to reduce the crack tip radius. Since the crack tip radius occurs in a square root form in A, such an effect may be assumed to be a second order effect and as such in this approximation may be neglected. Consequently we can assume that A is a constant and has the same value in biaxial as under uniaxial conditions. The variations of all the other terms may now be calculated.

σ_P for biaxial condition can be determined from equation (2.6)

Thus

$$\sigma_{PB} = \sigma_{PU}(1-\alpha+\alpha^2)^{-\frac{1}{2}} \qquad (3.1)$$

Similarly

$$E_B = E_U(1-\mu\alpha)^{-1} \qquad (3.2)$$

$$n_B = n_U \qquad (3.3)$$

$$\sigma_{yB} = \sigma_{yU}(1-\alpha+\alpha^2)^{-\frac{1}{2}} \tag{3.4}$$

$$C_B = C_u\left(1-\frac{\alpha}{2}\right)(1-\alpha+\alpha^2)^{\frac{n-1}{2}} \tag{3.5}$$

$$\left(\frac{\sigma_p}{\sigma_{yp}}\right)_B^{n-1} = \left(\frac{\sigma_p}{\sigma_{yp}}\right)_U^{n-1} \tag{3.5}$$

Substituting these relations in equation (1.19) we obtain

$$\sigma_{nB} = \frac{\sigma_{pU}(1-\alpha+\alpha^2)^{-\frac{1}{2}}+1}{1+\dfrac{A\lambda(1-\lambda)(1+0.595\lambda^2)}{1+\left\{\dfrac{1-\frac{\alpha}{2}}{1-\mu\alpha}\right\}(1-\alpha+\alpha^2)^{\frac{n}{2}}\left[C_n\dfrac{E}{\sigma_y}\cdot\left(\dfrac{\sigma_{pU}}{\sigma_{yp}}\right)^{n-1}\right]}} \tag{3.6}$$

Equation (3.6) is the governing equation to determine the biaxial failure from uniaxial failure. As an illustrative example let us consider the conditions for failure for the test specimens of the preceding example with $\alpha = 1/2$. We have

$$\sigma_{pU} = 78000\text{ psi},\quad C_n\frac{E}{\sigma_y}\left(\frac{\sigma_p}{\sigma_y}\right)^{n-1} = 180 \quad \text{and} \quad A = 2.82\times 180 = 508$$

$$(1-\alpha+\alpha^2)^{-\frac{1}{2}} = (.75)^{-\frac{1}{2}} = 1.154$$

$$\left\{\frac{1-\frac{\alpha}{2}}{1-\mu\alpha}\right\}(1-\alpha+\alpha^2)^{\frac{n}{2}} = \frac{(1-0.25)}{(1-0.3\times 0.5)}(1-0.5+0.25)^{\frac{9.53}{2}} = .229$$

$$\sigma_{nB} = \frac{\sigma_{pU}\times 1.154}{1+\dfrac{508\lambda(1-\lambda)(1+0.595\lambda)}{1+0.229\times 180}}$$

$$\therefore\quad \sigma_{nB} = \frac{1.154\times 78000}{1+12.0\lambda(1-\lambda)(1+0.595\lambda^2)}$$

$$\text{and}\quad \frac{\sigma_{nB}}{\sigma_{ul}} = \frac{1.25}{1+12.0\lambda(1-\lambda)(1+0.595\lambda^2)}$$

where σ_{ul} is the uniaxial ultimate tensile strength used here to non-dimensionally express the biaxial fracture strength. This curve for biaxial fracture and the corresponding uniaxial fracture curve are plotted in Fig.2 on a log-log paper. It will be observed from the figure that the effect of biaxiality is quite severe. In the range of λ between 0.1 and 0.4 the curve behaves essentially as a straight line. The fracture toughness index in this region is found to be 19900 psi. The slope of the curve is $-1/2.54$ and thus $a = 2.54$.

The trends predicted by the preceding analysis are essentially in line with the tests results for failure under biaxial conditions. Materials

under biaxial stress conditions tend to be somewhat more crack sensitive than under uniaxial conditions in the normal engineering test range of crack lengths. The steepness of the fracture curve is dependent upon the degree of biaxiality and the strain hardening coefficient. For materials with essentially flat top stress-strain curves, that is, as "n" takes on large values of about 50 or more, there are indications that the effect of biaxiality can become quite significant. From a study of equation (3.6) it will be found that for values of α between 0 and 1, the maximum decrease in fracture strength occurs for $\alpha \approx 0.5$.

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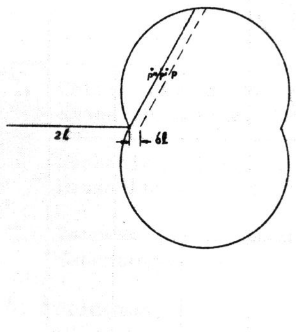

FIG.1

Schematic of the stress variation at a point p just ahead of the line of maximum principle stress. Stress at point p very close to stress at p' on the line of maximum principle stress. Stress at p" is on the elastic unloading part of the stress-strain curve.

FIG.2 PREDICTED UNIAXIAL AND BIAXIAL FRACTURE STRENGTHS IN THE PRESENCE OF THROUGH CRACKS IN PLANE SHEET SPECIMENS

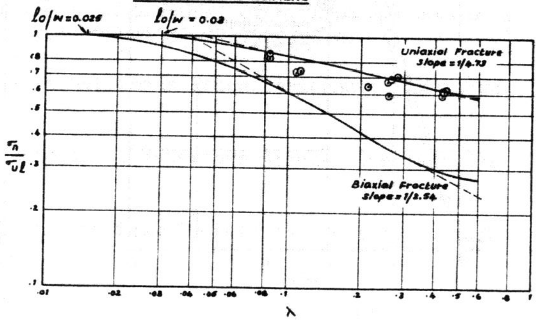

FIG.3

SCHEMATIC STRESS-STRAIN CURVE

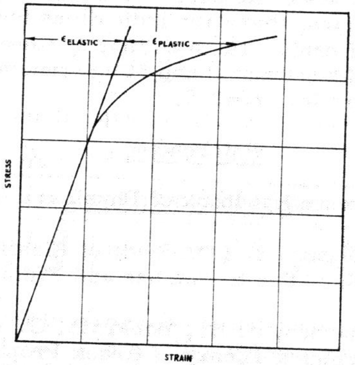

FIG.4 STRESS-STRAIN CURVE FOR A CYLINDER

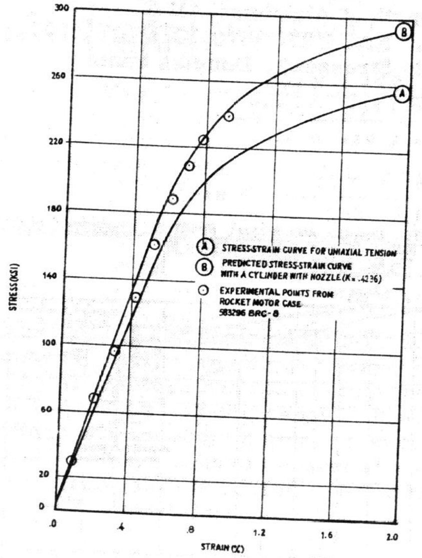

FIG.5 STRESS-STRAIN CURVE FOR A SPHERE