C-1 INITIATION AND GROWTH OF VISCOELASTIC FRACTURE

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ABSTRACT

Using an energy formulation including the work of viscous dissipation, fracture initiation and growth in linearly viscoelastic materials has been investigated. Assuming the viscoelastic character of the criterion is the same for all geometries as for the spherical cavity studied, the appropriate extension of the Griffith initiation criterion is believed established. Inasmuch as the threshold is time dependent, illustrative results are presented for four typical loading inputs, namely constant stress (strain), and constant stress rate (strain rate). If inertia effects are neglected, the formulation also permits a direct calculation of flaw size with time.

INTRODUCTION

The energy balance concept has been considered one of the cornerstones of fracture mechanics. Since its original application by Griffith(1) to brittle materials, Orowan(2) and Irwin(3) have applied it to ductile materials, and Rivlin and Thomas(4) to fracture in rubber. As summarized in an earlier review(5) one can take the position that the concept itself is independent of the material to which it is applied, providing that one express properly for each material the mechanism of energy dissipation during the fracture process.

For a conservative system, one needs therefore to separate the input energy into its various output components. In the simple brittle case, the deformations are all presumed to be elastic and the work put into the specimen is dissipated or transferred into the work of creating new surface and, for a propagating crack, into kinetic energy. In materials which are more rheologically complicated, as in the plastic, but time independent, flow of ductile materials or in the viscoelastic time dependent deformation of rubber, one quite clearly would expect these same dissipative mechanisms to exist, but in addition would also expect others which, respectively, include plastic work and viscous dissipation. Inasmuch as these quantities are all scalar, which

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is of course one of the main attractions of an energy approach to fracture, the previous "Griffith" critical stress results can characteristically be cast into similar form, e.g.

$$\sigma_{cr} = k\sqrt{ET_i/\ell} \qquad (1)$$

where k is a constant for the geometry under consideration, E the material modulus, and ℓ the crack length. In this expression T_i reflects the particular dissipation process for the material concerned; for brittle, ductile and viscoelastic ones respectively, one would tend to insert T_b, T_d or T_v. Qualitatively then the extension for a more general material would be written by replacing the scalar energies T_i by ΣT_i,

$$\sigma_{cr} = k\sqrt{(E/\ell)\ (T_b + T_d + T_v + \ldots)} \qquad (2)$$

which summation could also include T_{ke}, for the kinetic energy effect, along with any other appropriate terms. In brittle materials, for example, $T_b >> T_d + T_v + \ldots$, and one recovers the appropriate Griffith controlling dissipation term.

Whereas for ductile metals it has been shown that the plastic work contribution always exceeds the other terms numerically, the situation is not quite as simple for viscoelastic materials. Here, at cold temperatures the material is brittle and the T_b term dominates. At elevated temperatures however the viscous deformation associated with T_v controls. Quite clearly then at some intermediate temperature (or strain rate due to the WLF time-temperature association)[6] there is a point at which T_b and T_v are equally important, and therefore a generalized consideration such as (2) becomes attractive. Our experiments with rubber and plastic at various strain rate and temperatures have therefore led us to a more careful consideration of the dissipation mechanism in viscoelastic fracture. The major point for this discussion will be a calculation of the time dependent critical stress in a viscoelastic material subjected to various input works.

THERMODYNAMIC CONSIDERATIONS AND ENERGY PRINCIPLES

To the uninitiated, there has frequently been some confusion as to the manner by which the strain energy stored is transferred into surface energy, particularly insofar as the effect of stress or displacement boundary conditions is concerned. One convenient way of regarding this problem, which is related to the equivalent results obtained for fixed grip and fixed force fracture tests in elastic materials as indicated by Orowan[7], is to consider the appropriate variational energies [8], i.e. minimum potential energy, $V(u_i)$

$$V(u_i) = \int U\, d(vol) - \Sigma_{S_\sigma} \check{T}_i\, u_i \qquad (3)$$

where displacements are prescribed (with no body forces), and minimum complementary energy, $V^*(\sigma_{ij})$,

$$V^*(\sigma_{ij}) = \int U\, d(vol) - \Sigma_{S_u} T_i\, \check{u}_i \qquad (4)$$

where stresses are prescribed. The integrals are carried over S_σ and S_u, meaning the area over which stresses or displacements, respectively, are prescribed. In this way, when (3) and (4) are supplemented by the surface energy term $2\ell T$, which variation is always zero with respect to changes in applied stress or displacement unless crack elongation occurs, one can consider variations about the equilibrium state at which for a linear elastic system,*

$$\Sigma_{S_\sigma} T_i\, u_i|_{eq} = 2\, U_{eq} \qquad (5)$$

Thus one finds for example, upon varying the complementary energy at equilibrium with respect to crack length, holding the stress on the boundary fixed

$$\partial V^*_{eq}(\sigma_{ij})/\partial\ell = \partial U_{eq}|(\sigma_{ij})/\partial\ell - 2\partial U_{eq}(\sigma_{ij})/\partial\ell + 2T = 0$$

or

$$\partial U_{eq}(\sigma_{ij})/\partial\ell|_\sigma = 2T. \qquad (6)$$

Similarly, for the fixed grip loading holding u_i on the boundary fixed during the variation with crack length such that $\partial(\Sigma_{\sigma_u} T_i u_i)/\partial\ell = 0$

* If the load-deflection relation is non-linear, e.g. $p = k\varepsilon^n$, then the factor 2 is replaced by n + 1.

$$\partial V_{eq}(u_i)/\partial \ell = \partial U_{eq}(u_i)/\partial \ell + 2T = 0$$

or

$$-\partial U_{eq}(u_i)/\partial \ell|_u = 2T \qquad (7)$$

which are equivalent because

$$-\partial U_{eq}(\sigma_{ij})/\partial \ell|_\sigma = \partial U_{eq}(u_i)/\partial \ell|_u \qquad (8)$$

For viscoelastic materials exhibiting a rate dependence, it is believed more appropriate however to deal with the thermodynamic power equation [9,5] for the system, neglecting kinetic energy and body forces, which is

$$\dot{I} = \dot{F} + 2D + S\dot{E} \qquad (9)$$

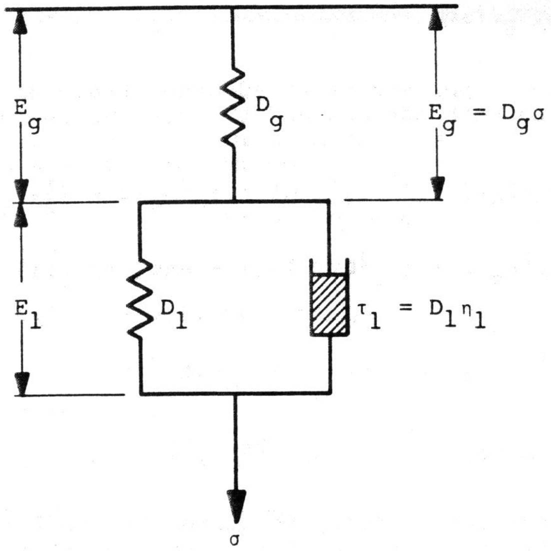
where the dot denotes differentiation with respect to time, and $\dot{I}$ = power input of the applied loading at the boundary, $\dot{F}$ = rate of increase of the free energy (strain energy), 2D = dissipation (mechanical power converted into heat flow) and $S\dot{E}$ = rate of increase of the surface energy. Specifically one has

$$\dot{I} = \int_{S_\sigma + S_e} \overset{\nu}{T}_i \, \dot{u}_i \qquad (10)$$

$$\dot{F} + 2D = \frac{d}{dt} \cdot \int_{vol} \int_o^t \sigma_i \dot{\varepsilon}_i \, dt \, d(vol) \qquad (11)$$

$$S\dot{E} = \frac{d}{dt} \int_s T \, d(surf) \qquad (12)$$

By way of simple illustration of the mechanics of the computation, consider the amount of strain energy and dissipation stored in a tensile specimen characterized by a simple three element viscoelastic body of a spring and Kelvin body in series (see insert),



Overall stress strain law: $\varepsilon = \varepsilon_g + \varepsilon_1$

$$\tau_1 \frac{d\varepsilon}{dt} + \varepsilon = D_g \tau_1 \frac{d\sigma}{dt} + (D_g + D_1)\sigma \qquad (13)$$

Stress distribution in two-element component

$$\sigma_{spring} = \varepsilon_1/D_1 \qquad (14)$$

$$\sigma_{dash} = \eta_1 \, d\varepsilon_1/dt \qquad (15)$$

For a step input, σ_o, in applied stress, one finds

$$\varepsilon_1 = D_1\sigma_o[1 - \exp(-t/\tau_1)] \qquad (16)$$

$$\sigma_s = \sigma_o[1 - \exp(-t/\tau_1)] \qquad (17)$$

$$\sigma_d = \sigma_o \exp(-t/\tau_1) \qquad (18)$$

and for the overall strain response

$$\varepsilon = \{D_g + D_1[1 - \exp(-t/\tau_1]\}\ \sigma_o \tag{19}$$

The direct approach is to add the strain energy per unit volume in both springs and the viscous energy in the dashpot.

$$U_{springs} = \sigma_o^2 D_g/2 + \int_o^t \sigma_s \dot{\varepsilon}_s\, dt = \sigma_o^2 D_g/2 + (\tfrac{1}{2} D_1) \int_o^t d\varepsilon_s^{\,2}$$

$$= \sigma_o^2 D_g/2 + (\sigma_o^2 D_1/2)\ [1 - \exp(-t/\tau_1)]^2 \tag{20}$$

$$U_{dashpot} = \int_o^t \sigma_d \dot{\varepsilon}_d\, dt = \eta_1 \int_o^t (d\varepsilon_d/dt)^2\, dt$$

$$= (\sigma_o^2 D_1/2)\ [1 - \exp(-2t/\tau_1)] \tag{21}$$

Hence the total input energy is stored in these two parts, one the elastic recoverable energy and the other the irrecoverable viscous work. Knauss[10] and Betz[11] have completed a study showing the relative percentage of work in each component for various strain-time histories. These results are particularly important in studying the effect and importance of strain history on time to failure. Whereas Smith[12] has proposed that at least for tensile specimens failure is rate independent, this conclusion should perhaps be restricted to monotonic loading histories where the dissipative contribution is a small percentage of the energy stored just prior to fracture.

The total stored energy can now be combined to yield

$$U_{springs} + U_{dashpot} = \sigma_o^2 D_1[1 - \exp(-t/\tau_1] + \sigma_o^2 D_g/2$$

$$= \sigma_o^2\{D_g + D_1[1 - \exp(-t/\tau_1)]\} - \sigma_o^2 D_g/2 \tag{22}$$

In the case of the generalized Kelvin model, i.e. an infinite combination of the two-element Kelvin bodies in series, the above expression is modified by replacing the subscript 1 by i, and summing over i. One further may note that because the general definition of the creep compliance is

$$D_{crp}(t) = D_g + \Sigma D_i[1 - \exp(-t/\tau_i)] \tag{23}$$

The previous result, (22), becomes, when generalized

$$U_{springs} + U_{dashpot} = \sigma_o^2[D_{crp}(t) - (D_g/2)] \tag{24}$$

The alternative approach is to use the definition, (11), considering a unit volume, to find

$$F + \int 2D dt = \int_o^t \sigma\dot{\varepsilon} dt = \int_o^t \sigma_o d[\sigma_o\{D_g + D_1[1-\exp(-t/\tau_1)]\}]/dt$$

$$= \sigma_o^2 D_g/2 + \sigma_o D_1[1-\exp(-t/\tau_1)] + \sigma_o^2 D_g/2 - \sigma_o^2 D_g/2$$

$$= \sigma_o^2\{D_g + D_1[1-\exp(-t/\tau_1)]\} - \sigma_o^2 D_g/2 \tag{25}$$

which is the same result.

As no creation of new surface is involved in this problem

$$\dot{F} + 2D = \sigma_o^2(D_1/\tau_1)\ \exp(-t/\tau_1) = \sigma\dot{\varepsilon} = \dot{I} \tag{25a}$$

At this point the difference between the energy balance in the small (per unit volume) illustrated above, and in the large can be recognized. In particular, the distribution of internal energy per unit volume is integrated up to time, t, then integrated over the volume, which in fracture problems can change with time as new surface is created, and then differentiated to balance the power input.

VISCOELASTIC FRACTURE

The calculations incident to fracture threshold are frequently algebraically complicated. In the original Griffith work on a stressed sheet containing a central crack, the degenerate solution from the Inglis elliptical hole was used, and the analysis conveniently required elliptic coordinates. It has been observed however that for a wide variety of discontinuity problems, the application of an energy balance criterion leads to essentially Griffith-type results which are quantitatively similar. For example the critical stress for a central crack of length 2a in a sheet is hardly different than that for a penny-shaped crack of radius a.[13] Upon comparing these two cases, and several others, one is tempted to inquire if there exists some other geometry which is sufficiently simple to demonstrate an energy instability, but yet tractable enough to study boundaries which may change with time.

Simplified Geometries

Of these classes, two are very appealing: (1) a cylindrical cavity, and (2) a spherical cavity. Consider for example the first of these, for which the stress distribution, independent of material properties, is

$$\sigma_{\binom{r}{\theta}}(r) = \sigma_o \frac{1 \mp (a^2/r^2)}{1 - (a^2/b^2)} \tag{26}$$

$$u(r) = \frac{\sigma_o r}{2E} \cdot \frac{1 + 3(a^2/r^2)}{1 - (a^2/b^2)} \; ; \; \nu = 1/2 \tag{27}$$

for internal radius a, external radius b, and uniformly loaded, $\sigma_r(b) = \sigma_o$. Using (5) for an assumed incompressible material,

$$U_{eq}(\sigma_{ij}) = (1/2)\, \sigma_o\, u(b) \cdot 2\pi b$$

$$= 2\pi \frac{\sigma_o^2 b^2}{4E} \cdot \frac{1 + (3a^2/b^2)}{1 - (a^2/b^2)} \tag{28}$$

So that using (6), and noting the surface energy on the inside radius is $2\pi aT$, such that $\partial S/\partial a = 2\pi T$

$$2\pi \frac{\sigma_o^2 b^2}{4E} \cdot \left[\frac{\partial}{\partial a} \frac{1 + (3a^2/b^2)}{1 - (a^2/b^2)}\right] = 2\pi T \tag{29}$$

and thus the critical applied stress at instability is

$$\frac{\sigma_{oc}}{1 - (a^2/b^2)} = \sqrt{\frac{ET}{2a}} \tag{30}$$

As a side issue which turns out to be important for other purposes, it may be observed that the maximum stress is at the internal radius,

$$\sigma_{max} = 2\sigma_{oc}/[1 - (a^2/b^2)] = 2\sqrt{ET/2a} \tag{31}$$

This result may be useful for determining T from small holes in pressurized membranes, or to connect the surface tension with maximum tensile stress for ring specimens ($a \to b$), as commonly used in certain tests of rubber ultimate properties[14] by replacing a by $\ell/2\pi$.

There is another interesting incidental observation. One can rearrange (31) to write

$$T = [\sigma_{max}^2/(2E)]a \equiv V_b a \tag{32}$$

Thomas[15], in his early work on the tearing of rubber, found that he obtained a reasonable approximation of the surface energy term, T, when he multiplied the uniaxial strain energy density at fracture by the local tear radius, which for a tiny cylindrical hole would be the hole radius a.

Without showing the calculations at this time because it will be done later in greater detail, the results for a spherical flaw in an incompressible body subjected to uniform tension, σ_o, at infinity are similar.

$$\frac{\sigma_{oc}}{1 - (a^3/b^3)} = \frac{4}{3}\sqrt{\frac{ET}{a}} \tag{33}$$

$$\sigma_{max} = 2\sqrt{ET/a} \tag{34}$$

which indicates that fracture in a spherical cavity occurs when the flaw size is twice as large as in a cylindrical cavity, presumably reflecting the increased biaxiality of the hoop stress and thus increased strength capability.

In summary then, the following table shows the critical applied stress for four cases

Critical Griffith Stresses

Geometry	2-D Crack (Griffith)	Cylindrical Cavity $a/b \to 0$	3-D Crack (Sneddon)	Spherical Cavity $a/b \to 0$
Critical Stress	$\sqrt{2/\pi}\sqrt{ET/a_o}$	$\sqrt{2}/2\sqrt{ET/a}$	$\sqrt{2\pi/3}\sqrt{ET/a_o}$	$(4/3)\sqrt{ET/a}$
Loading	Uniaxial	Biaxial	Uniaxial	Triaxial

Inasmuch as there is considerable quantitative similarity in the results, it is considered reasonable to use the simpler geometry and associated stress distributions for studying the fracture in viscoelastic media, where the difference would be expected to occur mainly in some modifications to the material (time dependent) modulus.

THE SPHERICAL CAVITY

The fracture instability threshold of a spherical cavity in an infinite incompressible medium subjected to uniform tension at infinity was treated earlier for the case of finite elastic deformations assuming a neo-Hookean material.(16) Before considering the case of a viscoelastic body and finite deformations however, it is pertinent to investigate the simple problem of infinitesimal deformations. The primary advantage in working with the spherical cavity, or for that matter, the cylindrical one, is that the stress distribution is independent of material properties. Hence one can use the viscoelastic analogy to determine with relative ease the viscoelastic strains or deformations, and thus proceed to calculate the various elements of the power equation.

Stress Loading

For a spherical cavity of initial radius a in a large medium where a uniform tension, $\sigma_o f(t)$, is applied at the radius b, one finds the stresses are given by

$$\sigma_r(r,t) = \sigma_o f(t) \frac{1 - \alpha(a_o^3/r^3)}{1 - \alpha k^3} \tag{35}$$

$$\sigma_\phi(r,t) = \sigma_\theta(r,t) = \sigma_o f(t) \frac{1 + (\alpha/2)\,(a_o^3/r^3)}{1 - \alpha k^3} \tag{36}$$

where

$$\alpha(t) \equiv [a(t)/a_o]^3 \quad ; \quad k = a_o/b \tag{37}$$

Using the Laplace transform analogy for the associated elastic problem,(17) one has

$$\bar{\sigma}_r(r,p) = \sigma_o L_r\left\{\frac{\sigma_r(r,t)}{\sigma_o}\right\} \tag{38}$$

$$\bar{\sigma}_\theta(r,p) = \sigma_o L_\theta\left\{\frac{\sigma_\theta(r,t)}{\sigma_o}\right\} \tag{39}$$

From the transformed stress-strain law for an incompressible medium ($\nu = 1/2$), $\varepsilon_\theta = \varepsilon_\phi = -\varepsilon_r/2$, there results

$$\bar{\varepsilon}_r(r,p) = D(p)\,[\bar{\sigma}_r - \bar{\sigma}_\theta] = p\bar{D}_{crp}(p)\,[\bar{\sigma}_r - \bar{\sigma}_\theta] \tag{40}$$

$$\bar{\varepsilon}_\phi(r,p) = \bar{\varepsilon}_\theta(r,p) = -D(p)\,[\bar{\sigma}_r - \bar{\sigma}_\theta]/2 = -p\bar{D}_{crp}(p)\,[\bar{\sigma}_r - \bar{\sigma}_\theta]/2 \tag{41}$$

where $\bar{D}_{crp}(p)$ is the transformed creep compliance, so that

$$\bar{\varepsilon}_r(r,p) = p\bar{D}_{crp}(p)\,\sigma_o[L_r - L_\theta]$$

$$= p\bar{D}_{crp}(p)\,\sigma_o L\left[-\frac{3}{2}\,\frac{\alpha(a_o^3/r^3)\,f(t)}{1 - \alpha k^3}\right] \tag{42}$$

$$\equiv \frac{\sigma_o}{r^3}\cdot p\bar{D}_{crp}(p)\,\bar{S}(p) \quad ; \quad S(t) \equiv -\frac{3}{2}\,\frac{\alpha a_o^3 f(t)}{1 - \alpha k^3}$$

and thus

$$\varepsilon_r(r,t) = \frac{\sigma_o}{r^3}\left\{\bar{D}_{crp}(o)\,S(t) + \int_o^t \frac{\partial D_{crp}(t-\tau)}{\partial(t-\tau)}\,S(\tau)\,d\tau\right\}$$

$$= -2\varepsilon_\theta(r,t) \tag{43}$$

where as a matter of definition, $D_{crp}(o) \equiv D_g = 1/E_g$.

Upon defining the instantaneous stored energy rate as

$$Q = \sigma_r\dot{\varepsilon}_r + \sigma_\theta\dot{\varepsilon}_\theta + \sigma_\phi\dot{\varepsilon}_\phi\Big|_{\nu=1/2} = (\sigma_r - \sigma_\theta)\,\dot{\varepsilon}_r \tag{44}$$

$$= \frac{\sigma_o^2}{r^6}\,S(t)\,\frac{\partial}{\partial t}\left\{D_g S(t) + \int_o^t \frac{\partial D_{crp}(t-\tau)}{\partial(t-\tau)}\,S(\tau)\,d\tau\right\} \tag{45}$$

Then

$$\int_o^t Q\,dt = \frac{\sigma_o^2}{r^6}\int_o^t S(\xi)\,\frac{\partial}{\partial\xi}\left[D_g S(\xi) + \int_o^\xi \frac{\partial D_{crp}(\xi-\tau)}{\partial(\xi-\tau)}\,S(\tau)\,d\tau\right]d\xi \tag{46}$$

and

$$\frac{1}{4\pi}\int_{a(t)}^b \int^t Q\,dt\;4\pi r^2\,dr = \frac{\sigma_o^2}{3}\,\frac{1 - k^3\alpha}{a_o^3\alpha}\int_o^t S(\xi)\,\frac{\partial}{\partial\xi}\left[D_g S(\xi) + \int_o^\xi \frac{\partial D_{crp}(\xi-\tau)}{\partial(\xi-\tau)}\,S(\tau)\,d\tau\right] \tag{47}$$

Finally, the energy stored is

$$\frac{1}{4\pi}\left[\dot{F} + 2D\right] = \frac{d}{dt}\frac{1}{4\pi}\int_{a(t)}^{b}\int_{o}^{t} Q\, 4\pi r^2 dt\, dr$$

$$= -\frac{\sigma_o^{\,2}}{2} f(t)\frac{\partial}{\partial t}\left[D_g S(t) + \int_o^t \frac{\partial D_{crp}(t-\tau)}{\partial(t-\tau)} S(\tau)d\tau\right]$$

$$+\frac{\sigma_o^{\,2}}{3a_o^{\,3}}\frac{\partial}{\partial t}\left[\frac{1-k^3\alpha}{\alpha}\right]\cdot\int_o^t S(\xi)\frac{\partial}{\partial\xi}\left[D_g S(\xi)\right.$$

$$\left.+\int_o^{\xi}\left[\frac{\partial D_{crp}(\xi-\tau)}{\partial(\xi-\tau)} S(\tau)d\tau\right]d\xi\right] \qquad (48)$$

where the definitions of $S(t)$ and $\alpha(t)$ have been incorporated.

The energy input due to the applied stress at the boundary $r = b$, is

$$\dot{I} = 4\pi\, b^2\, \sigma_o f(t)\; \dot{u}(b,t) = 4\pi\, b^3\, \sigma_o f(t)\; [-\dot{\varepsilon}_r(b)/2] \qquad (49)$$

so that

$$\dot{I}/4\pi = (-\sigma_o^{\,2}/2) f(t)\,\partial\left[D_g S(t) + \int_o^t \frac{\partial D_{crp}(t-\tau)}{\partial(t-\tau)} S(\tau)d\tau\right]/\partial t \qquad (50)$$

The surface energy rate is

$$\dot{SE}/4\pi = \partial(a^2T)/\partial t = 2a\dot{a}T \qquad (51)$$

Upon comparing the last expressions, it may be observed that the input rate term is balanced by the first of the two terms in the stored energy term so that the power equation finally reduces to

$$0 = \dot{a}\left[-\frac{\sigma_o^{\,2}}{a^4}\int_o^t S(\xi)\frac{\partial}{\partial\xi}\left[D_g S(\xi) + \int_o^{\xi}\frac{\partial D_{crp}(\xi-\tau)}{\partial(\xi-\tau)} S(\tau)d\tau\right]d\xi + 2aT\right] \qquad (52)$$

which is satisfied if the crack does not run, i.e. $\dot{a} = 0$.

On the other hand, another solution is seen to exist if the bracketted term vanishes. This condition is the integral relation for $a(t)$, which is the time history of the internal growing fractured surface, because

$$S(t) = S[a(t)] = -\frac{3}{2}\,\frac{a^3(t)\, f(t)}{1-[a(t)/b]^3} \qquad (53)$$

It is pertinent at this point to inquire into the physics of the condition because the variation with respect to time presumes the stress distribution stays symmetrical. It would seem somewhat unlikely that fracture would initiate simultaneously at an infinite number of uniformly distributed sources over the internal periphery of the flaw. Nevertheless, as argued earlier (16) in the paper dealing with finite elastic strains, it is at least a possible mechanism. For the purpose of investigating the form of the viscoelastic behavior therefore, it is also believed to be representative.

The general expression (52) is first written in the equivalent form

$$\frac{2a^5(t)T}{\sigma_o^{\,2}} = \frac{D_g}{2}\int_o^t \frac{d}{d\xi}\left[S^2(\xi)\right]d\xi + \int_o^t S(\xi)\left[\dot{D}_{crp}(o)\; S(\xi)\right.$$

$$\left.+\int_o^{\xi}\frac{\partial^2 D_{crp}(\xi-\tau)}{\partial\xi\,\partial(\xi-\tau)} S(\tau)d\tau\right]d\xi = \frac{D_g}{2}\left[\frac{3}{2}\,\frac{a^3(t)\; f(t)}{1-[a(t)/b]^3}\right]^2$$

$$+\dot{D}_{crp}(o)\int_o^t S^2(\xi)d\xi + \int_o^t\int_o^{\xi} S(\xi)\; S(\tau)\frac{\partial^2 D_{crp}(\xi-\tau)}{\partial\xi\,\partial(\xi-\tau)}\, d\tau\, d\xi$$

$$(54)$$

which can now be evaluated for special cases.

Step applied stress, $f(t) = 1(t)$

In this case the previous expression can be reduced, but for the present purposes consider the situation up to the time of fracture, $t = t_o$. During this period $a(t) = a_o$ and hence one finds

$$\frac{2a_o^5 T}{\sigma_o^2} = \left[-\frac{3}{2}\frac{a_o^3}{1-k^3}\right]^2 \left\{ \frac{D_g}{2} + \dot{D}_{crp}(o)t + \int_o^t \int_o^\xi \frac{\partial^2 D_{crp}(\xi-\tau)}{\partial\xi\,\partial(\xi-\tau)} d\tau d\xi \right\}$$

where the latter integral is

$$\int_o^t \int_o^\xi (\ldots)\, d\tau\, d\xi = -\dot{D}_{crp}(o)t + D_{crp}(t) - D_{crp}(o)$$

such that

$$\frac{2a_o^5 T}{\sigma_o^2} = \left[\frac{3}{2}\frac{a_o^3}{1-k^3}\right]^2 \{ D_{crp}(t_o) - (D_g/2)\}$$

where the similarity to the property dependence in (24) may be noted, and

$$\frac{\sigma_{oc}}{1-k^3} = \frac{4}{3}\sqrt{\frac{T/a_o}{2D_{crp}(t_o) - D_g}} \tag{55a}$$

The similar result for the viscoelastic cylinder is

$$\frac{\sigma_{oc}}{1-k^2} = \frac{2}{\sqrt{2}}\sqrt{\frac{T/a_o}{2D_{crp}(t_o) - D_g}} \tag{55b}$$

Note further that if the fracture is glassy and instantaneous such that $t_o \to 0$, $D_{crp}(t_o \to 0) = D_g = 1/E_g$ and from (55a)

$$\frac{\sigma_{oc}}{1-k^3} = \frac{4}{3}\sqrt{\frac{E_g T}{a_o}} \tag{56}$$

In the general case however, where σ_{oc} is imposed for a given flaw size, (55a) becomes the determining equation for the time to fracture. Given the material characterization in terms of the creep compliance, a curve is easily given for the fracture time (Figure 1,2).

The second quantity of interest is the initial velocity at fracture, $\dot{a}(t_o)$. In principle, if (52) were solved for $a(t)$ it would be a simple matter to determine the velocity and acceleration for all time, and also show among other things the time

at which kinetic energy effects would have to be considered in order to provide a limiting velocity. At the present time the author has been unable to find a suitable expression for the initial velocity and has not chosen to solve numerically. Relying on previous work however, (5,9) wherein an exponential crack growth was found, one may try a solution of the form

$$\frac{a(t)}{a_o} = e^{\lambda(t-t_o)}; \quad \frac{a(t_o)}{a_o} = \lambda \tag{57}$$

and substitute into the governing equation, expanding about the time t_o, to find

$$\frac{\dot{a}(t_o)}{a_o} = \frac{\dot{D}_{crp}(t_o)}{\frac{5}{3}[D_{crp}(t_o) - (D_g/2)] - D_g} \tag{58}$$

This approximation is not altogether satisfactory because for $t_o \to 0$

$$\frac{\dot{a}(o)}{a_o} = -\frac{6\,\dot{D}_{crp}(o)}{D_g} \tag{59}$$

whereas the velocity of crack propagation must be positive. It is even possible that the kinetic energy term if added would amount to a denominator addition of $D_g/6$ which would make the initial velocity infinite at the glassy condition. On the other hand for $t^* > t_o$ in the present expression such that $D_{crp}(t^*) = (11/10)D_g$ - only a ten percent increase in the small value of the glassy modulus, the initial velocity is high but finite, and decreases rapidly with increasing time to fracture. For example, for rubbery fracture controlled principally by $D_{crp}(t_o = t_e \to \infty) = D_e >> D_g$ (usually $D_e \approx 100\, D_g$),

$$\left.\frac{\sigma_{oc}}{1-k^3}\right|_e = \frac{4}{3}\sqrt{\frac{T/a_o}{2D_e - D_g}} = \frac{4}{3\sqrt{2}}\sqrt{\frac{E_e T}{a_o}} \tag{60}$$

with a velocity at fracture of

$$\frac{\dot{a}(t_e)}{a_o} \doteq \frac{3\dot{D}_{crp}(t_e)}{5D_e} \to 0 \quad \text{because } \dot{D}_{crp}(t_e) \to 0 \tag{61}$$

Some of these relations are illustrated in Figure 2.

Constant stress rate, $f(t) = k't$

For this loading condition, a similar calculation of the critical stress at failure, $\sigma_R = \sigma_o k' t_o$, or in terms of a prescribed stress rate R_σ, such that $\sigma_R = R_\sigma t$, yields an implicit relation from which to determine t_o

$$\frac{\sigma_{Rc}}{1-k^3} = \frac{R_\sigma t_o}{1-k^3} = \frac{4}{3}\sqrt{\frac{T/a_o}{(2/3)t_o \dot{D}_{crp}(o) + 2\int_o^1 \xi D_{crp}(\xi)d\xi}} \tag{62}$$

wherein the first moment of the creep compliance may be noted.

It is interesting to note that if one assumed the stress at failure, i.e. the circumferential stress on the inside surface of the spherical flaw, is the same regardless of whether it is reached by a step stress or ramp stress, then $\sigma_{oc} = \sigma_{Rc}$ and upon dividing (55b) by (62), it is easy to show that the time to failure in a constant stress rate test slightly exceeds the time to failure for a step stress, as expected.

Displacement Loading

Paralleling the elastostatic case of fixed grips, it is interesting to calculate the similar fracture instability criterion for a displacement loading at the outside boundary, $u_o(b,t)$. In this case, the incompressibility condition immediately yields,

$$u(r,t) = u_o(b,t)b^2/r^2 \equiv u_o g(t) b^2/r^2 \tag{63}$$

From the equilibrium relations one finds that the stresses are

$$\sigma_r(r,t) = \frac{4u_o}{3b}\left[\frac{b^3}{a^3} - \frac{b^3}{r^3}\right] L^{-1}\left[p\bar{E}_{rel}(p)\ \bar{g}(p)\right] \tag{64}$$

$$\sigma_\theta(r,t) = \frac{4u_o}{3b}\left[\frac{b^3}{a^3} + \frac{1}{2}\frac{b^3}{r^3}\right] L^{-1}\left[p\bar{E}_{rel}(p)\ \bar{g}(p)\right] \tag{65}$$

where $E_{rel}(t)$ is the relaxation modulus. In a similar fashion as before,

$$Q = (\sigma_r - \sigma_\theta)\dot{\varepsilon}_r = 4\left(\frac{u_o}{b}\right)^2\left(\frac{b}{r}\right)^6 \frac{\partial g(t)}{\partial t}\left\{E_g g(t) + \int_o^t \frac{\partial E_{rel}(t-\tau)}{\partial(t-\tau)} g(\tau)d\tau\right\} \tag{66}$$

from which the stored energy is found to be

$$\frac{d}{dt}\cdot\frac{1}{4\pi}\int_{a(t)}^b \int_o^t Q\, dt\, 4\pi r^2\, dr = -\frac{\dot{a}}{a^4}\int_o^t (r^6 Q)\, dt + \frac{1-k^3{}_\alpha}{3a_o{}^3{}_\alpha}[r^6 Q] \tag{67}$$

The energy input to the system is

$$\frac{1}{4\pi}\dot{I} = b^2\sigma_r(b,t)\,\dot{u}(b,t) = \frac{4b^3}{3}\left(\frac{u_o}{b}\right)^2\left(\frac{1-k^3{}_\alpha}{k^3{}_\alpha}\right)\frac{\partial g(t)}{\partial t} \times \left\{E_g g(t) + \int_o^t \frac{\partial E_{rel}(t-\tau)}{\partial(t-\tau)} g(\tau)d\tau\right\} \tag{68}$$

while the surface energy rate is

$$\dot{S}E/4\pi = 2a\dot{a}T \tag{69}$$

Upon writing the power equation, (9), one finds

$$\dot{a}\left\{2aT - \frac{4b^6}{a^4}\left(\frac{u_o}{b}\right)^2\int_o^t \frac{\partial g}{\partial\xi}\left[E_g g(\xi) + \int_o^\xi \frac{\partial E_{rel}(\xi-\tau)}{\partial(\xi-\tau)} g(\tau)d\tau\right]d\xi\right\} = 0 \tag{70}$$

which again has the stationary solution, $\dot{a} = 0$, plus an additional one.

Constant strain input, $g(t) = 1(t)$

In this case the work input is due to the elastic term, after which the stresses relax to lower than their maximum glassy values. The integration yields, for $\dot{a} \neq 0$

$$2aT = \frac{4b^4}{a^4}\int_o^{o^+}\frac{\partial}{\partial\xi}[u(b,\xi)]\left\{L^{-1}[E(p)\ \bar{u}(b,p)]\right\}d\xi + \frac{4b^4}{a^4}\int_{o^+}^t(\ldots.)d\xi \tag{71}$$

During the first integral $E(p) \rightarrow E_g$, i.e. the glassy value, and during the second integral $\partial u(b,\xi)/\partial\xi = 0$ leaving

$$2aT = \frac{4b^4}{a^4}\int_0^{0^+}(E_g/2)\, d[u^2(b,\xi)] = \frac{4b^4}{a^4}\cdot\frac{E_g u_o^2}{2}$$

from which

$$\left(\frac{u_o}{b}\right)^2 = \frac{T/a}{E_g}\cdot\left(\frac{a}{b}\right)^6 \qquad (72a)$$

or, at the flaw

$$\varepsilon_\phi^{\,2}(a,t) = E_g^{-1}T/a \qquad (72b)$$

and because u_o is a constant, $a(t) = a_o$. Thus if during a constant strain test the applied displacement u_o is not sufficient to impose immediately the critical value corresponding to the glassy modulus, stress relaxation will occur and fracture can not result.

Constant strain rate, $\varepsilon = R_\varepsilon t;\ u(b,t) = bR_\varepsilon t \equiv u_o t$

Here the second integral does contribute to give

$$2aT = \frac{4b^4}{a^4}\cdot\left(\frac{u_o}{b}\right)^2\int_0^t\left[E_g\xi + \int_0^\xi \frac{\partial E_{rel}(\xi-\tau)}{\partial(\xi-\tau)}\,\tau\, d\tau\right]d\xi$$

$$= \frac{4b^4}{a^4}\left(\frac{u_o}{b}\right)^2\int_0^t\int_0^\xi E_{rel}(\tau)d\tau\, d\xi \equiv \frac{4b^4}{a^4}\left(\frac{u_o}{b}\right)^2 E_{rel}^{(2)}(t) \qquad (73)$$

from which

$$\left(\frac{u_o}{b}\right)^2 = R_\varepsilon^2 = \frac{T/a}{2}\cdot\frac{(a/b)^6}{E_{rel}^{(2)}(t)} = \frac{(T/a_o)k^6}{2E_{rel}^{(2)}(t)}\cdot\left(\frac{a(t)}{a_o}\right)^5 \qquad (74)$$

At the fracture time $t = t_o$, for a prescribed imposed strain rate, the relation between fracture time and flaw size is

$$R_\varepsilon^2 = \frac{(T/a_o)k^6}{2E_{rel}^{(2)}(t_o)} \qquad (75)$$

or in terms of the boundary displacement $u_R(b,t_o) = bR_\varepsilon t_o$, or boundary strain,

$$\varepsilon_\theta^{\,2}(b) = \left(\frac{u_r}{b}\right)^2 = \frac{T/a_o}{2}\cdot\frac{k^6}{t_o^{-2}\,E_{rel}^{(2)}(t_o)} \qquad (76a)$$

or alternatively in terms of the strain at the flaw

$$\varepsilon_\theta^{\,2}(a) = \left[\frac{b^3}{a^3}\,\varepsilon_\theta(b)\right]^2 = \frac{T/a_o}{2}\cdot\frac{1}{t_o^{-2}\,E_{rel}^{(2)}(t_o)} \qquad (76b)$$

where one may note by dividing (76) by (72),

$$\left[\frac{u_R(b)}{u_\varepsilon(b)}\right]^2 = \frac{E_g\,t_o^2/2}{E_{rel}^{(2)}(t_o)} \geqslant 1 \qquad (77)$$

because $E_{rel}^{(2)}(t_o) \leqslant \int_0^{t_o}\int_0^\xi E_g d\tau\, d\xi = E_g t_o^{\,2}/2$. Hence the applied displacement to failure in a constant strain rate test always exceeds that for constant strain, as expected.

Finally when the fracture commences, it will progress proportional to the first integral of the relaxation modulus, or what is equivalent*, to the tensile modulus as measured in a constant strain rate test.(17)

$$\left[\frac{a(t)}{a_o}\right]^5 = \frac{2R_\varepsilon^2}{(T/a_o)k^6}\,E_{rel}^{(2)}(t)\ ;\quad t \geqslant t_o$$

$$\frac{\dot a(t)}{a_o} = \frac{2R_\varepsilon^2}{5k^6T/a_o}\cdot\frac{a_o^4}{a^4}\int_0^t E_{rel}(\xi)d\xi = \varepsilon_\theta^2(a,t)\left[\frac{a(t)}{a_o}\right]^2\frac{2t^{-2}E_{rel}^{(1)}(t)}{5T/a_o} \qquad (78)$$

and the initial velocity can be expressed

$$\frac{\dot a(t_o)}{a_o} = \frac{1}{5t_o}\cdot\frac{t_o^{-1}\,E_{rel}^{(1)}(t_o)}{t_o^{-2}\,E_{rel}^{(2)}(t_o)} \qquad (79)$$

* Note, $R_\varepsilon\int_0^t E_{rel}(\tau)d\tau \equiv \sigma_{tens}(t)$.

Furthermore, one can calculate the circumferential stress at the flaw, using (65), to find at failure ($t = t_o$)

$$\sigma_\theta(a,t_o) = \sqrt{\frac{T/a_o}{2} \cdot \frac{2E_{rel}^{(1)}(t_o)}{\sqrt{E_{rel}^{(2)}(t_o)}}} \tag{80}$$

This critical stress has been calculated as a function of strain rate for the Solithane 113 material with the result shown in Figure 3, which also incorporates the uniaxial tensile failure stress at constant rate for reference. While there is no reason for them to be identical - because of the different stress states, it may be observed that the shapes of the curves are qualitatively similar, with the gross failure probably occurring at longer times (smaller R_ε) than that for the initial flaw to enlarge.

MODIFICATIONS FOR FINITE STRAINS

It was previously mentioned that a finite elastic analysis for a neo-Hookean material had been completed.(16) Since then, the analysis has been extended to include a Mooney-Rivlin material.(18) The neo-Hookean results are shown in Figure 4. It is natural to inquire if the character of the viscoelastic behavior will change due to finite strain. Unfortunately this analysis is not yet practical. Nevertheless, some indications of expected results can be conjectured from the work of Smith(19) and Guth et al(20) on tensile specimens. It was found that for many materials the time and strain dependence separated, with the time-temperature WLF shift factor, still applying. Denoting now the stretch ratio by λ, ($\lambda = 1 + \varepsilon$ for small strains), it was found that

$$\sigma(\lambda,t) = E_{rel}(t)\, f(\lambda) \tag{81}$$

which sensibly permits an ad hoc extension of the previous finite elastic results to include viscoelastic effects by replacing the material modulus in the finite elastic strain analysis by its appropriate (infinitesimal) analog. The extent to which this idea may apply to three-dimensional geometries is currently being investigated more completely, both theoretically and experimentally.

EXPERIMENTAL RESULTS

An experimental apparatus to generate hydrostatic tension in a rubber "poker-chip" specimen was described earlier.(5,21)

Basically a thin clear rubber disk was cemented flatwise between two clear relatively rigid plastic heads and pulled perpendicular to its faces. The cement bond restrained contraction perpendicular to the line of pull, and developed a lateral cross-tension which, in the vicinity of the incompressible rubber specimen, was equal to the longitudinal tensile stress. Arrangement was made to observe any events in the specimen by inserting 45 degree mirrors in the loading heads and providing for photographic recording.

Two results are of concern to this discussion. First, small fracture loci are observed to appear suddenly as the specimen is loaded. Post fracture analysis has led us to conclude that micro-origins of the order of 10^{-4} inch exist in the specimen prior to loading and that fracture has initiated from one of these latent flaws. Furthermore, from independent measurements of the surface energy, T, and use of the theoretical finite strain predictions for the Mooney-Rivlin type material, which adequately represents our Solithane 113 material,(22) it can be inferred from the applied load at fracture as observed during the poker-chip test, that the initial radius of the flaw to which fracture was subsequently traced, a_o, should have been of the order of 10^{-4} inch. Figure 5 shows a photograph of the fracture surface and probable origin.

Second, when the specimen was loaded at essentially constant strain rate, photographs were taken of the flaw growth. The predictions based upon the infinitesimal viscoelastic deformation analysis (78) have been compared with the test data. The variations of the cavity size with time are qualitatively similar, both having negative curvature at small times, but the range of times over which the growth was observed, of the order of fractions of a second before the size was influenced by the proximity of the specimen edge, was too small to obtain quantitative results at this time. The sensitive dependence of the cavity size upon the fifth power of the second integral of the relaxation modulus indicates that it may be impractical to attempt much more than initial velocity or acceleration measurements by the use of this specimen.

CONCLUDING REMARKS

The major purpose of this paper has been to deduce an extended Griffith criterion for the fracture of linearly viscoelastic media. By use of the thermodynamic power equation it proves possible also to deduce the time history of the fracture. While the fracture models chosen are not perhaps those commonly associated with crack-type fracture, the similarity in results among all elastic critical stress values has led the author to believe that the use of a uniformly growing spherical or cylindrical cavity will lead to representative results, particularly insofar as the viscoelastic dependence is concerned.

Contrary to the initial expectations, the result did not lead to a different value for T_v, but to the value T_b except whereas previously a constant modulus was used in brittle fracture, it is now proposed that a time-dependent modulus, which must be associated with the loading history, be used. For example, a step function in applied stress is expected to lead to replacing the constant compliance in the usual Griffith formulas by $2D_{crp}(t_o) - D_g$, as shown for the two cases of the cylindrical and spherical flaws.

This conclusion gives rise to an interesting fundamental question: Is the surface tension, T, rate dependent? Discussions with physical chemists reveal that this matter is not resolved on the phenomenological basis. It is particular pertinent because Thomas and Greensmith(23) have shown that the energy change is rate dependent in their experiments. Nevertheless their results can probably be slightly reinterpreted to show in the Griffith sense that the product ET is rate dependent, with however the rate dependency arising from the material modulus and a constant T, rather than the inverse which is basically their implication. As far as the analysis herein is concerned it is a simple matter to allow for the surface energy to vary, T = T(t), if appropriate.

The results presented do not specifically extend the analysis to different temperatures, as was implied in the earlier discussion of brittle versus viscous behavior at low and high temperatures, θ, respectively. The time-temperature superposition principle, however, permits such extension if the material representation is expressed in terms of the temperature reduced time, $t_R = t/a_\theta(\theta)$.

In conclusion it is hoped that it will be possible to verify, at least within engineering accuracy, that the viscoelastic correction for finite strain which is mandatory for very small flaws can be applied in the ad hoc fashion mentioned herein by utilizing the finite elastic analysis already completed and replacing the constant material modulus constants by the same viscoelastic equivalents as in the case of small strains. And finally, it will be essential to verify if the viscoelastic fracture criterion based on cylindrical and spherical flaws will hold for crack-type fracture. Such experiments in cracked sheets have already been initiated, but the results are not available at the present time.

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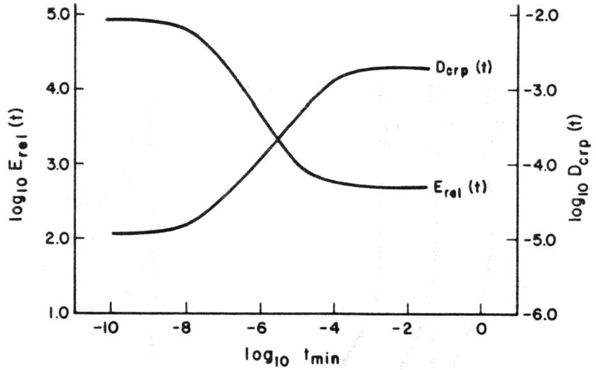


FIG. 1. Material property characterization for Solithane 113 at 20°C

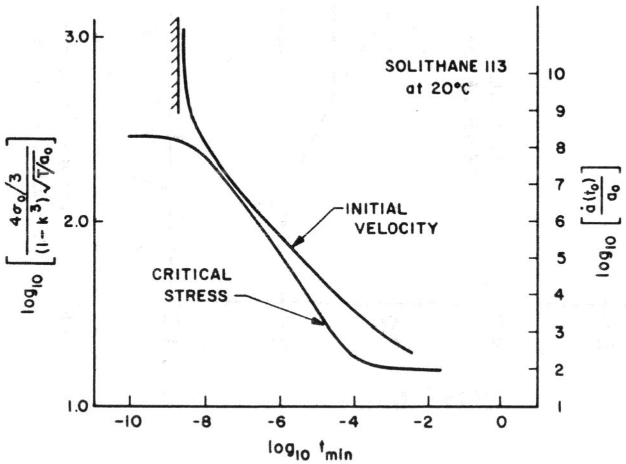


FIG. 2. Time to failure and initial fracture speed for a spherical flaw under constant stress

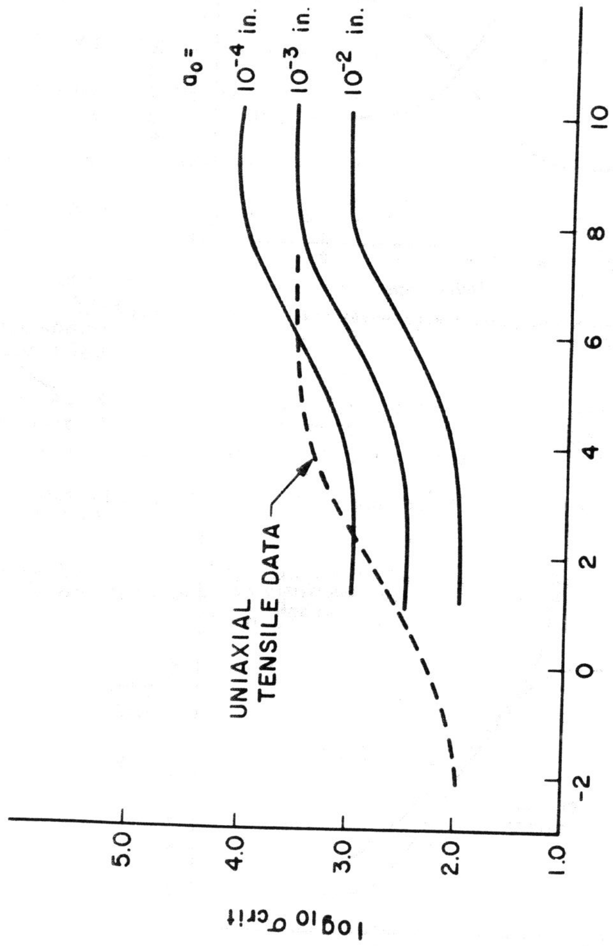


FIG. 3. Failure stress (psi) at the spherical flaw for various initial sizes ($T = 0.05$ lbs/in.) in Solithane 113 at 20°C.

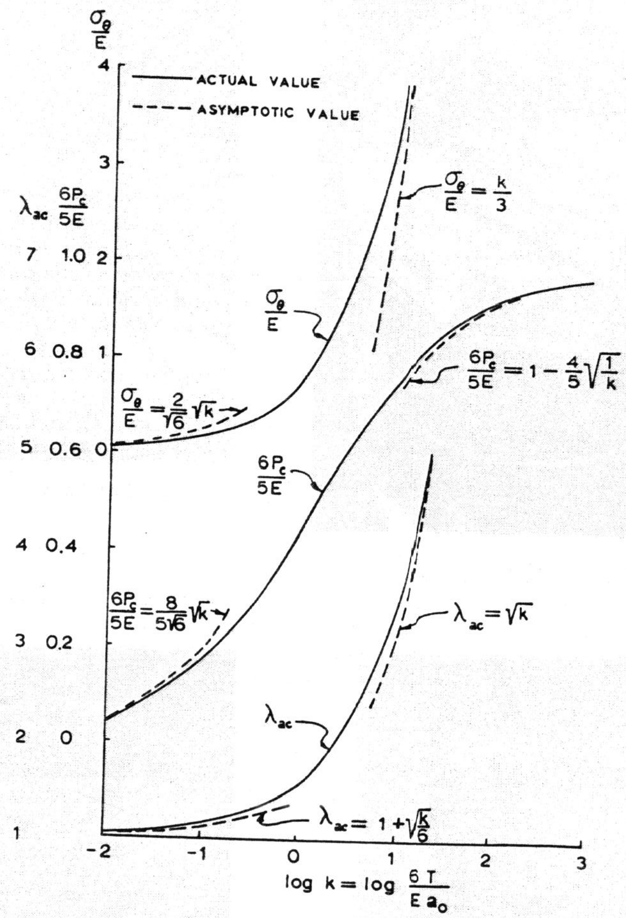
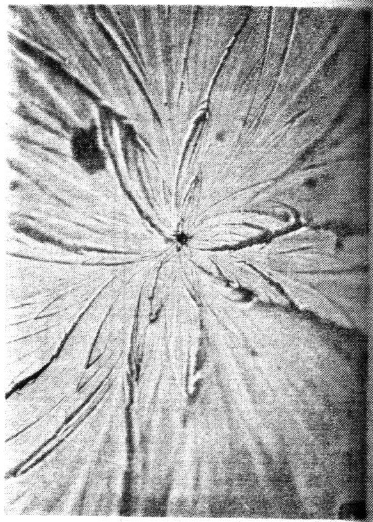
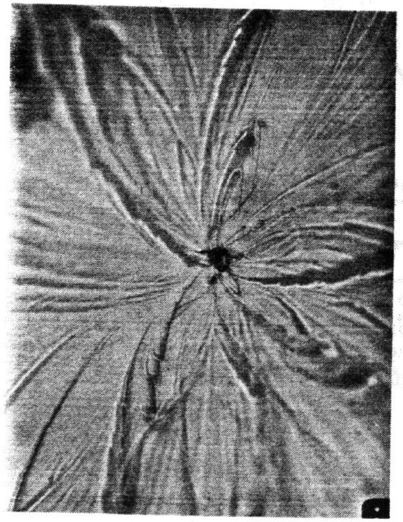


Figure 4. Critical conditions for instability of an initial spherical flaw allowing for large strains ($\lambda_a = a/a_o$). W (Neo-Hookean) $= E(I_1 - 3)/6 = E(2\lambda_\theta^2 + \lambda_\theta^{-4} - 3)/6$.

(a)Fracture surface



(b) 86 X



(c) 170 X

(d) 680 X

Figure 5. Fracture under hydrostatic tensione and the fracture nucleus at increasing magnifications.[24]