A-7

The Interactions of Cracks and Dislocations
I. Screw Dislocations

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Abstract

The forces acting on screw type dislocations in the neighborhood of a two dimensional crack are evaluated taking due account of the effect of the dislocation on the crack. From a consideration of these forces, expressions are derived for the number of dislocations which can be caused to enter the crack and for the energy of a crack-dislocation pair. The latter are obtained only under certain simplifying circumstances. Finally, the effect of the presence of dislocations on crack propagation is considered.

1. Introduction

It is very widely held that the stresses produced in the neighborhood of a crack under load can be such as to cause dislocations to move so as to interfere with the motion of the crack. The objective of the present paper is to examine quantitatively such elastic interactions of cracks and dislocations.

To facilitate the treatment the problem is treated as one in two dimensions.

2. Process of Analysis

We suppose the crack to be of infinite extent in the z-direction, to lie in the plane $y = 0$ and the range of x given by $-c \leq x \leq c$. We suppose also there exist a dislocation which lies parallel to the z-axis and which intersects the plane $z = 0$ at the point $x = \eta$, $y = \xi$.

The crack and dislocation will interact. In effect we evaluate this interaction by first determining the displacements of the crack surfaces which result from the presence of the stress field of the dislocation. We then note that any given displacement of the crack surfaces has components U_x, U_y, and U_z and that these components may be represented (1) by three noninteracting distributions of elementary dislocation. Two of these distributions are of edge character with displacement vectors severally in and perpendicular to the plane of the crack. They relate to the displacements U_x and U_y respectively. The third is of screw type and is related to U_z. Given the form and dislocation character of each distribution present the energy of and the stress field due to the crack can be calculated.

Here, we are given in particular the stress $\bar{\sigma}_i(x_0)$ due to the dislocation at a point (η, ξ) and have first to find the distribution functions $f_i(x)$, $(i = 1,2,3)$.

In these terms the conditions for equilibrium of the crack surface are

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written

$$A \int_{-c}^{*c} f_i(x) \frac{dx}{x - x_o} = \bar{\sigma}_i(x_o) \qquad (1)^*$$

Here $A = \frac{\mu\lambda_i}{2\pi k}$, μ is the shear modulus, λ_i is the displacement vector of a unit positive dislocation. $k = 1 - \nu$ and $k = 1$ when the distributions are of edge and screw character respectively. ν is Poisson's ratio.

If, additionally, we suppose the existence of some uniform loading with stress components σ_i equation (1) becomes

$$A \int_{-c}^{*c} \frac{f_i(x)\, dx}{x - x_o} = \bar{\sigma}_i(x_o) + \sigma_i \qquad (2)^*$$

The solution to this equation is [2,3]

$$f_i(x) = \frac{-1}{\pi^2 A \sqrt{(c^2-x^2)}} \int_{-c}^{*c} \sqrt{(c^2-t^2)}(\bar{\sigma}_i(t) + \sigma_i) \frac{dt}{t-x} + Q/\sqrt{(c^2-x^2)} \qquad (3)^*$$

Where Q is a constant to be determined from the condition that

$$\int_{-c}^{c} f_i(x)\, dx = n_i$$

Where n_i is such that $n_i\lambda_i$ is the total dislocation content of the distribution $f_i(x)$.

Given $f(x)$, the stress fields of the crack may be evaluated through integrals of the form

$$I(x_o, y, c) = \int_{-c}^{c} f(x)\, G(x, x_o, y)\, dx \qquad (4)$$

Where $G(x,x_o,y)$ is a known function. In the next few subsections we apply this analysis to determine the forces acting on a dislocation in the neighborhood of a crack.

3. Forces on Screw Dislocations

Since the study of the more complicated and rather more interesting cases of edge dislocations is not yet sufficiently complete, we shall, in the remainder of this paper, restrict our considerations to dislocations of screw type.

In accord with the definition that λ_i be positive we define a positive stress to be one which tends to move a positive dislocation in the direction of increasing x. On this basis the stress $\sigma(t)$ due to a screw dislocation

* Principal value is to be taken.

of Burgers vector λ located at $x = \eta$, $y = \xi$ is

$$\frac{\mu\lambda}{2\pi} (t-\eta) \Big/ \left((t-\eta)^2 + \xi^2\right)$$

Substituting this value in equation (3) and integrating we have

$$f(x) = \frac{1}{\sqrt{(c^2-x^2)}} \left\{\frac{\sigma x}{\pi A} + Q - \frac{\lambda}{\pi\lambda_i}\left[\frac{R}{r}\sin\left(\phi + \frac{\theta}{2}\right) - 1\right]\right\} \qquad (5)$$

where,

$$R = \left\{\left((c-\eta)^2 + \xi^2\right)\left((c+\eta)^2 + \xi^2\right)\right\}^{1/4}$$

$$r = \left((\eta - x)^2 + \xi^2\right)^{1/2}$$

$$\theta = \arctan 2\eta\xi/(\eta^2-\xi^2-c^2)$$

$$\omega = \arctan (\eta-x)/\xi$$

$$\lambda/\lambda_i = \pm 1$$

We find that the number of dislocations each of Burgers vector λ_n contained in the crack is

$$n = \int_{-c}^{c} f(x)\, dx = \pi Q$$

so that $Q = n/\pi$

The distribution function is then

$$f(x) = \frac{1}{\pi\sqrt{(c^2-x^2)}} \left\{\frac{\sigma x}{A} + \frac{1}{\lambda_i}\left[n\lambda_n + \lambda - \lambda\frac{R}{r}\sin\left(\phi + \frac{\theta}{2}\right)\right]\right\} \qquad (6)$$

where $\lambda_n/\lambda_i = \pm 1$

The stresses at any point x,y are obtained through Equation (4) on giving G appropriate values. We have

$$\sigma_{xz} = \int_{-c}^{c} f(t) \frac{\eta y}{(t-x)^2 + y^2} dt \qquad (7)$$

and

$$\sigma_{yz} = \int_{-c}^{c} f(t) \frac{x-t}{(t-x)^2 + y^2} dt$$

The evaluation of these integrals is straightforward. In particular the stresses induced at the dislocation are

$$\sigma_{yz} = \frac{\sigma}{R}\left\{\eta \cos\frac{\theta}{2} + \xi \sin\frac{\theta}{2}\right\} - \sigma$$

$$- \frac{A\lambda}{2\lambda_i}\left\{\frac{\sin\theta}{2\xi} + \frac{1}{R^2}\left[\eta \cos\theta + \xi \sin\theta\right]\right\}$$

$$+ \frac{A}{\lambda_i}\,\frac{(n\lambda_n + \lambda)}{R}\cos\frac{\theta}{2} \qquad (8)$$

$$\sigma_{xz} = \frac{\sigma}{R}\left\{\eta \sin\frac{\theta}{2} - \xi \cos\frac{\theta}{2}\right\}$$

$$- \frac{A\lambda}{2\lambda_i}\left\{\frac{\sin^2\theta/2}{\xi} - \frac{1}{R^2}\left[\xi \cos\theta - \eta \sin\theta\right]\right\}$$

$$+ \frac{A}{\lambda i}\,\frac{(n\lambda_n + \lambda)\sin\frac{\theta}{2}}{R} \qquad (9)$$

The forces acting on the dislocation due to the crack has components

$$F_x = \lambda\,\sigma_{yz}$$

$$F_y = \lambda\,\sigma_{xz}$$

We now examine the values of these forces at particular values of μ and ξ for positive $(\lambda/\lambda_i = 1)$ and negative $(\lambda/\lambda_i = -1)$ dislocations.

3.1. $\eta = -a,\ \xi = 0,\ \lambda/\lambda_i = 1,\ \lambda_n/\lambda_i = 1$

From Equations (7) and (8) we have

$$F_y = 0$$

and

$$F_x = \frac{\lambda}{\sqrt{(a^2-c^2)}}\left\{\sigma a - A(n+1) + \frac{Aa}{\sqrt{(a^2 - c^2)}}\right\} - \sigma\lambda \qquad (10)$$

Adding to this, the force $\sigma\lambda$ due to the applied stress field σ we have a total force

$$F_t = \frac{\lambda}{\sqrt{(a^2-c^2)}}\left\{\sigma a - A(n+1) + \frac{Aa}{\sqrt{(a^2-c^2)}}\right\} \qquad (11)$$

We see that when $n = 0$ and $\sigma \geq 0$, F_t is positive for all a and we note that the existence of a positive F_t implies that the dislocation will tend

to enter the crack.

It is clear by inspection that, for a given c and a positive σ, $F_t(a) = F_t$ will pass through a minimum. If $F_t \geq 0$ at this minimum it will be positive everywhere. On the basis that $F_t = 0$ at the minimum we find that the crack will continute to attract dislocations if

$$\frac{\sigma c}{A}\left(1 + \left(\frac{\sigma c}{A}\right)^{-2/3}\right)^{3/2} \geq n + 1 = n_m \qquad (12)$$

For n large this approximates to

$$\frac{\sigma c}{A} + \frac{3}{2}\left(\frac{\sigma c}{A}\right)^{\frac{1}{3}} \geq n + 1 \qquad (13)$$

To illustrate we take $\sigma/\mu = 1/400$, $2c = 1$ micron $\simeq 4 \times 10^3\lambda$ and find the maximum value of n + 1 to be

$$n_m \simeq 35$$

By way of comparison we note that the no. of dislocations in a pile-up of length 2c is $\sigma c/A$ which here ~30.

3.2. $\eta = -a,\ \xi = o,\ \lambda/\lambda_i = -1,\ \lambda_n/\lambda_i = 1$

Here

$$F_y = 0$$

$$F_x = \frac{-\lambda}{\sqrt{(a^2-c^2)}}\left\{\sigma a - A(n-1) - \frac{Aa}{\sqrt{(a^2-c^2)}}\right\} + \sigma\lambda \qquad (14)$$

and

$$F_t = \frac{-\lambda}{\sqrt{(a^2-c^2)}}\left\{\sigma a - A(n-1) - \frac{Aa}{\sqrt{(a^2-c^2)}}\right\} \qquad (15)$$

F_t is positive for all a if $\sigma = 0$ or $n = 0$. For $\sigma > 0$ and for integral values of n, F_t is greatest for a = c, decreases with increasing values of "a" becoming negative. Accordingly, negative dislocations will tend to enter the crack only from positions comparatively close to it.

3.3 $\eta = a,\ \xi = 0,\ \lambda/\lambda_i = -1,\ \lambda_n/\lambda_i = 1$

$$F_y = 0$$

$$F_t = \frac{-\lambda}{\sqrt{(a^2-c^2)}}\left\{\sigma a + A(n-1) + \frac{Aa}{\sqrt{(a^2-c^2)}}\right\} \qquad (16)$$

For $\lambda_n/\lambda_i = 1$, F_t is negative for all a and for $\sigma \geq 0$. For $\lambda_n/\lambda_i = -1$,

F_t is the negative of the value given by equation (10). The considerations of Paragraph 3.1 for positive dislocations are thus also relevant for negative dislocations.

3.4. $\eta = a$, $\xi = 0$ $\lambda/\lambda_i = 1$, $\lambda_n/\lambda_i = 1$

$$F_y = 0$$

and

$$F_t = \frac{\lambda}{\sqrt{(a^2-c^2)}} \left\{ \sigma a + A(n+1) - \frac{Aa}{\sqrt{(a^2-c^2)}} \right\} \qquad (17)$$

F_t is largest in magnitude and negative when $a \sim c$, with increasing "a" it decreases in magnitude becoming positive, and the remarks of Section 3.2 are valid here.

It is pertinent to inquire whether dislocations inserted in the crack at one end according to the considerations of Section 3.1, can escape at the other.

A pre-requisite for escape is that F should become positive when $a = c + \gamma\lambda$, where γ is of order unity. Assigning this value to a, Equation (17) becomes on setting $n + 1 = n_m = \frac{\sigma c}{A}$

$$F = \sqrt{\frac{\lambda}{2c\gamma}} \left\{ 2\sigma c - A \frac{c}{2\gamma\lambda} \right\} = 0 \qquad (18)$$

For the case we have already considered viz: $\sigma/\mu = 1/400$, $c = 2 \times 10^3 \lambda$, we find $\gamma = 1/4$; a result which might suggest that escape would occur. However, we shall find later (Sect. 4.2) that the energy of a dislocation at such a point exceeds that of a "dislocation" within the crack by an amount $2\sigma c\lambda$. Treating γ as variable it is clear from inspection of Equation (18) that this energy cannot be provided through the operation of the force F through one or a few distances, λ.

We conclude then that escape cannot occur, at least in the absence of thermal assistance.

3.5. $\eta = 0$, $\xi = y$, $\lambda/\lambda_i = \pm 1$

We have from (7) and (8)

$$F_t = \sigma\lambda|y|/\lambda_i \sqrt{(y^2 + c^2)}$$

$$F_y = - \frac{Ay}{2\lambda_i^2 |y|} \left\{ \frac{\lambda}{|y|} + \frac{\lambda|y|}{y^2+c^2} - \frac{2(n\lambda_n+\lambda)}{\sqrt{(y^2+c^2)}} \right\} \qquad (19)$$

We observe that F_t is independent of n and F_y of σ. Further, we see that as $c \to \infty$, $F_y \to -A\lambda/2y$, independent of the sign of the dislocation. We note that for y positive, dislocations are attracted to the crack whilst F_y is negative and thus, if λ and λ_n are of the same sign, for finite c and

arbitrary y only when $n = 0$.

Dislocations (of the same sign) at positions $\eta = 0, \xi = \pm y_k$ will tend to enter the crack, if the $|y_k| \ll c$, to a number

$$m = c/2|y_m|$$

where $|y_m|$ is the largest of the $|y_k|$.

3.6. General Case

For the sake of completeness we now set down expressions for the distribution function, stress and force components for the general case in which the crack contains n positive dislocations, and m dislocations of either sign are located at the points η_j, $\xi_j (j = 1,2,\ldots m)$ in its vicinity. The distribution is

$$f(x) = \frac{1}{\pi\sqrt{(c^2-x^2)}} \left\{ \frac{\sigma x}{A} + n - \sum_{j=1}^{m} \frac{\lambda_j}{\lambda_i} \left(\frac{R_j}{r_j} \sin\left(\phi_j + \frac{\theta_j}{2}\right) - 1 \right) \right\} \qquad (20)$$

Where

$$R_j = \left\{ (\overline{c - \eta_j}^2 + \xi_j^2)\ (\overline{c + \eta_j}^2 + \xi_j) \right\}^{1/4}$$

$$\theta_j = \arctan 2\eta\ \xi_j/(\eta_j^2 - \xi_j^2 - c^2)$$

$$\phi = \arctan (\eta_j - x)/\xi_j$$

$$\gamma_j = (\overline{\eta_j - x}^2 + \xi_j^2)^{1/2}$$

$$\sigma_{yz} = \frac{\sigma}{T}\{\alpha x + \beta y\} - \sigma + \frac{A\alpha}{T}\left\{ n + \sum_{j=1}^{m} \lambda_j/\lambda_i \right\} + A \sum_{j=1}^{m} \frac{\lambda_j}{\lambda_i} H_j\ (x,y) \qquad (21)$$

$$\sigma_{xz} = \frac{\sigma}{T}\{\beta x - \alpha y\} + \frac{A}{T}\beta\left\{ n + \sum_{j=1}^{m} \lambda_j/\lambda_i \right\} + A \sum_{j=1}^{m} \frac{\lambda_j}{\lambda_i} F_j\ (x,y) \qquad (22)$$

$$\text{Where } H_j(x,y) = -\frac{1}{S_j}\left\{ \overline{x - \eta_j} \left\{ \overline{\eta_j - x}^2 + \xi^2 + y^2 \right\} \right.$$

$$+ \frac{R_j}{T}\left[\alpha\left\{ \left(\alpha_j \overline{y_j - x} + \beta_j\xi_j\right)\left(\overline{\eta_j - x}^2 - y^2 + \xi^2\right) \right.\right.$$

$$\left.\left.\left. - 2\alpha_j y^2 \overline{x - \eta_j} \right\} + \beta_y \left\{ \alpha_j \left(\overline{\eta_j - x}^2 + y^2 - \xi_j^2 \right) - 2\xi_j\ \overline{x - y_j} \right\} \right] \right\} \qquad (23)$$

$$F_j(x,y) = -\frac{1}{S_j}\Big[y\{\overline{\eta_j - x}^2 - \xi_j^2 + y^2\}$$

$$+\frac{R_j}{T}\{(\overline{x-\eta_j}^2 - y^2 + \xi^2)(\alpha_j\alpha\, y + \alpha_j\beta\,\overline{\eta_j - x} + \beta\beta_j\xi_j)$$

$$+ 2(x-\eta_j)\, y\,(\alpha\alpha_j\,\eta_j - x + \alpha\beta_j\,\xi_j - \alpha_j\beta y)\}\Big] \qquad (24)$$

and

$$S_j = \left(\overline{\eta_j - x}^2 + \overline{\xi_j - y}^2\right)\left(\overline{\eta_j - x}^2 + \overline{\xi_j + y}^2\right),$$

$$\alpha = \cos \phi/_2,\ \beta = \sin \phi/_2$$

$$\phi = \arctan 2xy/(x^2 - y^2 - c^2),$$

$$\alpha_j = \cos \theta_j/2,\ \beta_j = \sin \theta_j/2,$$

$$T = \left\{(\overline{x - c}^2 + y^2)\ (\overline{x + c}^2 + y^2)\right\}^{1/4}.$$

From these, the components of force on the t^{th} dislocation due to the presence of the crack are:

$$F_x = \lambda_t \Big\{\frac{\sigma}{R_t}(\alpha_t\eta_t + \beta_t\xi_t) - \sigma + \frac{Aa_t}{R_t}\Big(n + \sum_{j=1}^{m}\lambda_j/\lambda_i\Big)\alpha_t$$

$$-\frac{A}{2}\Big\{\frac{\alpha_t\beta_t}{\xi_t} + \frac{1}{R_t^2}\Big[\eta_t\,\overline{\alpha_t^2 - \beta_t^2} + 2\alpha_t\,\beta_t\,\xi_t\Big]$$

$$+ A\sum_{j\neq t}^{m}\frac{\lambda_j}{\lambda_i}G_j(\eta_t, \xi_t) \qquad (25)$$

$$F_y = \lambda_t \Big\{\frac{\sigma}{R_t}\big(\beta_t\,\eta_t - \alpha_t\xi_t\big)$$

$$-\frac{A}{2}\Big\{\frac{\beta_t^2}{\xi_t} - \frac{1}{R_t^2}\Big[\xi_t\,\overline{\alpha_t^2 - \beta_t^2} - 2\alpha_t\,\beta_t\,\eta_t\Big]\Big\}$$

$$+\frac{A}{R_t}\beta_t\Big(n + \sum_{j=1}^{m}\frac{\lambda_j}{\lambda_i}\beta_t\Big) + A\sum_{j\neq t}^{m}\frac{\lambda_j}{\lambda_i}F_j(\eta_t, \xi)\Big\} \qquad (26)$$

We shall employ these relations to examine some facets of dislocations-crack interaction when $m = 2$.

We suppose that n dislocations have entered the crack and that n has the limiting value n_m defined by Equation (12). We inquire; can an additional dislocation be caused to enter as a consequence of the presence of another dislocation supposed fixed in position.

We consider two cases: (a) the fixed dislocation is positive, and (b) it is negative.

3.7. Fixed Dislocation-Positive

We take the moving dislocation to be at $-a$, 0 and the other to be fixed at η_2, ξ_2. Including the effect of the applied stress and that due to the direct interaction of the dislocations:

$$-A\ \overline{\eta_2 + a}\ /\ (\overline{\eta_2 + a}^2 + \xi_2^2),$$

we have from Equation (25)

$$F_t = \lambda\Big\{\frac{\sigma}{R_1}a - \frac{A(n_m + 2)}{R_1} + \frac{Aa}{R_1^2} - \frac{A}{S_2}\Big[-(a + \eta_2)(\overline{\eta_2 + a}^2 + \xi_2^2)$$

$$-\frac{R_2}{R_1}(\alpha\ \overline{\eta_2 + a} + \xi_2\beta_2)(\overline{\eta_2 + a}^2 + \xi_2^2)\Big]\Big\} \qquad (27)$$

where $R_1 = \surd(a^2 - c^2)$

$$R_2 = \left\{(\overline{\eta_2 - c}^2 + \xi_2^2)(\overline{\eta_2 + c}^2 + \xi_2^2)\right\}^{1/4}$$

$$S_2 = (\overline{\eta_2 + a}^2 + \xi_2^2)^2$$

Writing $\psi = \arctan \xi_2/\overline{a+\eta_2}$ Equation (27) becomes

$$F_t = \frac{\lambda}{\surd(a^2-c^2)}\Big\{\sigma a - A(n_m + 2) + \frac{Aa}{\surd(a^2-c^2)}$$

$$+ A\frac{R_2}{S_2^{1/4}}\cos(\psi - \theta/2)\Big\} \qquad (28)$$

Referring to Equation (11) we see that F_t will be positive and consequently that the dislocation will tend to enter the crack if

$$(\overline{\eta_2 - c}^2 + \xi_2^2)^{1/4}(\overline{\eta_2 + c}^2 + \xi_2^2)^{1/4}\cos(\psi - \tfrac{\theta}{2}) > 2(\overline{\eta_2 + a}^2 + \xi^2)^{1/2}$$

It is clear from inspection that this inequality can be satisfied only for η negative. Furthermore, from the geometrical description of the angle $\psi = \theta/2$ given in Fig. 1, it is clear that the entry of a dislocation which "passes" the static dislocation is debarred.

In general then the presence of a single, or as is readily seen, any number of positive dislocations will not assist the entry of another positive dislocation into the crack.

3.8. Fixed Dislocation Negative

We again take the mobile dislocation to be at -a, 0 and the fixed dislocation to be at η_2, ξ_2. Analogously, with Equation (27) we have

$$F_t = \frac{\lambda}{\sqrt{(a^2-c^2)}}\left\{\sigma a - An_m + \frac{Aa}{\sqrt{(a^2-c^2)}} - A\frac{R_2}{S^{1/4}}\cos\overline{\psi - \theta/2}\right\} \tag{29}$$

The dislocation will tend to enter if

$$\frac{R_2}{S^{1/4}}\cos\psi - \theta/2 < 0$$

and thus if $|\psi - \theta/2| > \frac{\pi}{2}$.

To decide this question it is simplest to resort to geometry. It is found that the angle θ is given by the relation

$$\theta = \chi + \phi$$

where χ and ϕ are as defined in Fig. 1. A little geometry then gives the angle $(\frac{\theta}{2} - \psi)$ as shown. It is clear by inspection that this angle can exceed $\frac{\pi}{2}$ only when η_2 is negative and then only for a limited range of a. In general then the presence of a single negative dislocation will not assist the entry of an additional positive dislocation into the crack.

If we now suppose that two negative dislocations are sited, for simplicity, at the equivalent positions η_2, ξ_2; η_2 $-\xi_2$ Equation (29) becomes

$$F_t = \frac{\lambda}{\sqrt{(a^2-c^2)}}\left\{\sigma a - A\,n_m + \frac{Aa}{\sqrt{(a^2-c^2)}} + A - 2A\frac{R_2}{S^{1/4}}\cos\overline{\psi - \theta/2}\right\}$$

The positive dislocation will tend to enter the crack if

$$S^{1/4} - 2R_2\cos(\psi - \theta/2) > 0 \tag{30}$$

This condition is satisfied for a considerable range of η. For example, with $\eta=0$, (30) becomes

$$\sqrt{(a^2 + \xi^2)} > 2\sqrt{(c^2 + \xi^2)}\cos(\psi - \theta/2)$$

Reference to Figure 1 shows that this inequality is satisfied for all a if, at a = c, $\psi - \theta/2 > \pi/3$, which is certainly attainable. Again, we note that the $n_m + 1^{th}$ dislocation having been added, the addition of yet another negative dislocation will permit the entry of an additional positive and so on, provided that for the $n_m + N^{th}$ dislocation

$$S^{1/4} - \sum_{j=1}^{N} R_j\cos(\psi_j - \frac{\theta_j}{2}) > 0$$

Finally, we observe that if the number of negative dislocations be N + p, then the number, N, of positive dislocations can be increased without limit if p is unlimited.

4. Crack Energy

We now consider the effect on crack energy of the presence of screw dislocations. The energy of a crack under a stress σ containing n dislocations and with m dislocations in its vicinity may be obtained either from the forces and displacements in the plane of the crack or from the forces acting on the dislocations as they are brought one by one from some distant point to the positions which they are to occupy.

4.1. Energy when m = 0

We obtain an energy(4)

$$E_n = \frac{n^2\mu\lambda^2}{4\pi}\ln\frac{2R}{c} - nc\lambda\sigma - \frac{\pi\sigma^2c^2}{\mu} + 4\,Sc \tag{31}$$

where 4Sc is the energy of the crack surfaces.

4.2. Energy when m = 1, Special Cases

4.2.1 Positive or Negative Dislocation at $\eta = -a$, $\xi = 0$.

Employing Equation (11) and integrating we find that the total energy of crack and dislocation is

$$E = E_n + \sigma\lambda\left(\sqrt{(a^2 - c^2)} - a\right)$$

$$+ \frac{\mu\lambda^3}{4\pi\lambda_i}\left\{(2n + \frac{\lambda}{\lambda_i})\ln\frac{2R}{a + \sqrt{(a^2 - c^2)}} + \frac{\lambda}{\lambda_i}\ln\frac{2(a^2 - c^2)}{r_o(a + \sqrt{(a^2-c^2)})}\right\}; \tag{32}$$

the self energy of the dislocation at -a, 0 being $\frac{\mu\lambda^2}{4\pi}\ln\frac{R}{r_o}$.

When $\frac{a-c}{c} = \frac{\delta}{c} \ll 1$ this becomes

$$E \simeq E_n - \sigma\lambda c + \frac{\mu\lambda^3}{4\pi\lambda_i}\left\{(2n + \frac{\lambda}{\lambda_i})\ln\frac{2R}{C} + \ln\frac{4\delta}{r_o}\right\} \tag{33}$$

This compatible with (31) if $\delta = r_o/4$.

4.2.2. Positive or Negative Dislocation at $\eta = 0, \xi = +d$

Integrating the term F_y of Equation (18) we obtain

$$E = E_n + \frac{\mu\lambda^3}{4\pi\lambda_i}\left\{(2n + \frac{\lambda}{\lambda_i})\ln\frac{2R}{|d|+\sqrt{(c^2+d^2)}} + \frac{\lambda}{\lambda_i}\ln\frac{2d\sqrt{(c^2+d^2)}}{(|d|+\sqrt{(c^2+d^2)})r_o}\right\} \quad (34)$$

For $\frac{d}{c} = \frac{\delta}{c} << 1$ this becomes

$$E - E_n + \frac{\mu\lambda^3}{4\pi\lambda_i}\left\{(2n + \frac{\lambda}{\lambda_i})\ln\frac{2R}{c} + \ln\frac{2\delta}{r_o}\right\} \quad (35)$$

4.2.3 Positive or Negative Dislocation at $\eta = a, \xi = 0$

We obtain from equations (16) and (17)

$$E = E_n + \sigma\lambda\,(a - \sqrt{(a^2-c^2)})$$

$$+ \frac{\mu\lambda^2}{4\pi}\left\{(2n + \frac{\lambda}{\lambda_i})\ln\frac{2R}{a + \sqrt{(a^2-c^2)}} + \frac{\lambda}{\lambda_i}\ln\frac{2(a^2-c^2)}{r_o(a + \sqrt{(a^2-c^2)})}\right\} \quad (36)$$

When $(a-c)/c = \frac{\delta}{c} << 1$ this becomes

$$E = E_n + \sigma\lambda c + \frac{\mu\lambda^3}{4\pi\lambda_i}\left\{(2n + \frac{\lambda}{\lambda_i})\ln\frac{2R}{c} + \ln\frac{4\delta}{r_o}\right\} \quad (37)$$

4.3. Term in $\sigma\lambda c$

We see that the term $\sigma\lambda c$ is taken negatively in (33), where $\eta = -c$, and positively in (37), where $\eta = c$. It is absent in (35), $\eta = 0$. These differences are representative of the facts that while no work is done in moving a dislocation from -c to c,nevertheless the work done in taking the dislocation from $-R_\infty$ to R_∞ must be $2\sigma\lambda R_\infty$.

4.4. Energy when $m \geq 1$, Dislocations at Arbitrary Positions

The energy for this, the general case has not been evaluated since the integrals involved are seemingly intractable and because expressions for this energy contribution are not essential for our later considerations of crack propagation.

5. Crack Propagation

We have seen that in the absence of dislocation pile-ups and other stress concentrating effects the largest number of dislocations which may enter the crack is

$$n_m \simeq \sigma c/A$$

unless dislocations of the opposite sign to those within the crack are present in the neighborhood of the crack in considerable number. We have also seen how crack energy is affected by the presence of dislocations. We now employ these results to determine the effects of dislocations, in and around the crack, on the Griffith condition for crack propagation.

5.1. Crack Containing n_m Dislocations

According to equation 31 the energy of the crack is

$$E = \frac{n_m^2\mu\lambda^2}{4\pi}\ln\frac{2R}{c} - n_m c\lambda\sigma - \pi\frac{\sigma^2c^2}{\mu} + 4Sc$$

The Griffith condition requires that for crack propagation

$$\frac{\partial E}{\partial c}_{(n_m)} \leq 0$$

Differentiating and setting $n_m = \sigma c/A$ we have

$$\sigma_c \geq \sqrt{\frac{\mu S}{\pi c}} \quad (39)$$

The stress σ_c so defined is just half that required when the n_m dislocations are absent.

5.2. Propagation Condition for the General Case: Dislocation at η, ξ

As we have remarked in Section 4.4, it is difficult to calculate crack-dislocation energy when the dislocation is at arbitrary η, ξ. Nevertheless, we can evaluate the effect of such dislocations on crack propagation by setting the work done on an element $f(x)\ \delta x$ of the array through a stress $\tau(x)$ when the crack increases in length from 2c to 2 $(c + \delta c)$ as

$$\delta w = \lambda\ \tau(x)\ f(x)\ \frac{x}{c}\ \delta\ c$$

This step is justified if, denoting the distribution function for the new crack length by $\bar{f}(x)$, we have $f(\frac{x}{c}) \equiv \bar{f}\ (\frac{x}{c+\delta c})$. Here this is true in the limit as $\delta c/c \to 0$. For the whole distribution we have

$$\lim_{\delta c \to 0} \frac{\delta w}{\delta c} \equiv -\frac{dE'}{dc} = \frac{\lambda}{c}\int_{-c}^{c} xf(x)\ \tau(x)\ dx \quad (40)$$

Substituting for $f(x)$ from Equation 6 with $\lambda_n/\lambda_i = 1$ and setting $\tau(x) = \sigma + A\ \overline{x-\eta}/(\overline{x-\eta}^2 + \xi^2)$ we have

$$\frac{dE'}{dc} = \frac{\lambda}{c}\int_{-c}^{c}\left(\sigma + A\,\frac{\overline{x-\eta}}{\overline{x-\eta}^2 + \xi^2}\,\frac{x}{\sqrt{(c^2-x^2)}}\left\{\frac{\sigma x}{A} + \left(n + \frac{\lambda}{\lambda_i}\right)\right.\right.$$

$$\left.\left. - \frac{\lambda}{\lambda_i} R\,\frac{(\alpha\,\overline{\eta-x} + \beta\xi)}{\overline{\eta-x}^2 + \xi^2}\right\}\right) dx \qquad (41)$$

Integrating and adding the quantities $4S$ and $-n^2\,\mu\lambda^2/4\pi - n\sigma\lambda$, which arise respectively from the crack surface energy and the self energy of the dislocations within the crack, we have the propagation condition,

$$4S - \frac{n^2\mu\lambda^2}{4\pi c} - n\sigma\lambda = \frac{\lambda}{c}\left\{\frac{\sigma^2c^2}{2A} + \frac{\sigma}{R}\frac{\lambda}{\lambda_i}\left[\alpha(R^2 - \eta^2 + \xi^2) - 2\lambda\eta\xi\right]\right.$$

$$+ \frac{A}{R}\frac{\lambda}{\lambda_i}\left(n + \frac{\lambda}{\lambda_I}\right)\left[R - \alpha\eta - \beta\xi\right]$$

$$\left. - \frac{A}{2}\left[1 + \alpha^2 - \frac{\alpha\beta\eta}{\xi} - \frac{1}{R^2}\left\{\eta^2 - \xi^2\cos\theta + 2\eta\xi\sin\theta\right\}\right]\right\} \leq 0 \qquad (42)$$

The implications of this result are not easily assessed from a direct consideration of the expression. Accordingly we shall consider some special cases.

5.3. Dislocation at $\eta = -a$, $\xi = 0$

We obtain either from Equation (42) or by differentiating Equation (33) the condition

$$\varepsilon_n + \varepsilon_1 \leq 0$$

where

$$\varepsilon_n = -\frac{n^2\mu\lambda^2}{4\pi c} - n\sigma\lambda - \frac{\pi\sigma^2 c}{\mu} + 4S \qquad (44)$$

and

$$\varepsilon_1 = -\frac{\lambda}{c}\sqrt{(a^2-c^2)}\,\frac{\lambda}{\lambda_i}\left[\sigma c^2 + A\left(n + \frac{\lambda}{\lambda_i}\right)\left(a - \sqrt{(a^2-c^2)}\right) + \frac{\lambda}{\lambda_i}\frac{Ac^2}{\sqrt{(a^2-c^2)}}\right] \qquad (45)$$

For the case $n = n_m \gg 1$, ε_1 is dominated by the term in n which is positive for λ/λ_i negative and vice versa. When $n = 0$, all terms are negative when λ/λ_i is positive and all are negative save that in σ when λ/λ_i is negative.

5.4. Dislocation at $\eta = a$, $\xi = 0$

ε_n and ε_1 are identical with the values found in Section 5.3 save that the sign of the coefficient of σ in ε_1 is changed. Thus, for this situation

also propagation will be assisted by the presence of a positive and hindered by a negative dislocation for the case $n \gg 1$. Similarly, for $n=0$ the presence of a dislocation of either sign will tend to aid propagation.

5.5. Dislocation at $\eta = 0$, $\xi = d$

ε_n is again as defined in Equation (44)

$$\varepsilon_1 = \frac{\mu\lambda^3}{4\pi\lambda_i}\sqrt{(c^2 + d^2)}\left\{-2\left(n + \frac{\lambda}{\lambda_i}\right)\frac{c}{d + \sqrt{c^2+d^2}} + \frac{\lambda}{\lambda_i}\frac{c}{\sqrt{c^2+d^2}}\right\}$$

As before with n large, ε_1 is dominated by the term in n and is thus negative for positive λ/λ_i and vice versa. For $n = 0$, ε_1 is negative independent of the sign of λ/λ_i.

5.6 Two Negative Dislocations at $\eta = 0$, $\xi = \pm d$

We have seen in Sect. 3.8 that the two negative dislocations may be positioned near the crack in such a manner as to permit the entry of another positive dislocation into the crack.

We suppose for simplicity that the negative dislocations are located at $\eta = 0$, $\xi = d$; $\eta = 0$, $\xi = -d$. Allowing that an additional positive dislocation has entered the crack we have a propagation condition

$$4S = \mu\,\frac{\overline{n+1}^2\,\lambda^2}{4\pi c} - \overline{n+1}\,\sigma\lambda + \frac{\mu\lambda^2}{2\pi\sqrt{(c^2+d^2)}}\left\{2(n-1)\frac{c}{d + \sqrt{(c^2+d^2)}} + \frac{c}{\sqrt{c^2+d^2}}\right\} \leq 0 \qquad (47)$$

For propagation to be rendered easier by the addition of these three dislocations we require

$$-\sigma\lambda - \frac{\mu\lambda^2}{4\pi}\left\{\frac{2n+1}{c} - \frac{4(n-1)\,c}{\sqrt{(c^2+d^2)}\,(d + \sqrt{(c^2+d^2)})} - \frac{2c}{\sqrt{c^2+d^2}}\right\} < 0$$

which is so, for n large, if $-1 + 2\left(1 - \frac{d}{\sqrt{(d^2+c^2)}}\right) < 0$ which implies $d > c/\sqrt{3}$. For this case reference to Fig. 1 shows that $\frac{\theta}{2} - \phi > \frac{\pi}{3}$. These results prompt the conjecture that crack propagation cannot be rendered easier by increasing n_m through the addition of negative dislocations in the neighborhood of the crack in any way whatever (cf Sect. 3.8).

6. Summary and Conclusions

We have found:

(1) A crack empty of dislocations attracts dislocations.

(2) The number (n_m) of dislocations which can be inserted in a crack through the presence of a uniform shear stress somewhat exceeds the number in a simple pile up of the same length.

(3) Dislocations so inserted at one end of a crack cannot readily escape at the other.

(4) In general, a crack containing n_m dislocations repels dislocations of the same sign and attracts those of the opposite sign to those within it.

(5) In general, the energy of a crack-dislocation combination is greater than their combined energy at infinite separation if the force on the dislocation is repulsive.

(6) The tendency for crack propagation increases as the number n of dislocations within it increases. When $n = n_m$ the stress required is rather less than half that required when $n = 0$.

(7) When n is not zero, the tendency for crack propagation is increased by the presence of dislocations of the same sign as those within the crack.

(8) When $n = 0$ this tendency is increased by the presence of a preponderance of dislocations of the same sign. Accordingly, in this case the movement of screw dislocations of a given sign towards or into a propagating crack will not hinder its motion.

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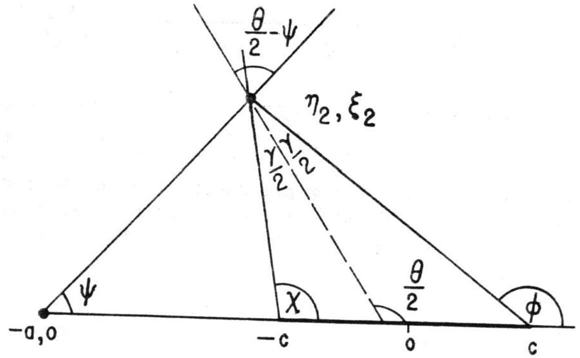

FIG.1--THE DESCRIPTION OF THE ANGLES $\psi, \frac{\theta}{2}$ AND $\frac{\theta}{2} - \psi$.

THE CRACK LIES BETWEEN -C AND C; SURFACE DISPLACEMENTS ARE NORMAL TO THE PLANE OF THE PAPER.