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THE INFLUENCE OF ELASTIC ANISOTROPY ON THE PROPAGATION OF FRACTURE

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ABSTRACT

The model of a propagating crack introduced by Craggs for an isotropic solid is extended to the case of general elastic anisotropy. The general theory for propagation under tensile or shear stresses is derived. As for most two dimensional problems in anisotropic elasticity the solution involves the roots of a sixth degree polynomial so that it is necessary to proceed numerically at some stages. A computor program has been written to do this. This is used to show that the shape of the square cracks which are produced in the interior of silicon-iron by the internal pressure of electrolytic hydrogen may be due to elastic anisotropy. On this basis, predictions can be made as to the shape of cracks in other metals, in particular molybdenum, vanadium and tantalum.

1. INTRODUCTION

In this paper the model of a propagating crack which was introduced by Craggs (1) for the case of an elastically isotropic medium is extended to general anisotropy. A partial extension has been given by Atkinson (2) for the case where the crack front is perpendicular to a symmetry plane of the medium. This is the greatest degree of anisotropy for which it is possible to obtain explicit solutions but it is still very restrictive, e.g. in a crystal of cubic symmetry the crack front must be along a crystallographic direction either of type ⟨100⟩ or ⟨110⟩. In the analysis given here for general anisotropy(§ 2-7) it will be noted that at an early stage (equation (6)) we require the roots of a sextic polynomial. Since these cannot be obtained explicitly, any application of this general theory must of necessity be numerical. However it is reasonably straight forward to program the necessary computations for a computor. A Fortran program was developed to obtain the numerical results presented in § 9 where this theory is used to explain the shape of cracks produced during the hydrogen charging of silicon-iron (3) and predictions are made of the shapes which might be produced if the experiment were repeated on other metals.

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2. MATHEMATICAL SPECIFICATION

Following Craggs (1) we consider an infinite elastic solid with a semi-infinite plane crack propagating through it with constant speed v_1. The crack extends over the half plane $x_2 = 0$, $x_1 < v_1 t$ where t is the time. The crack front is parallel to the x_3 axis and the elastic state of the body is independent of x_3. The crack is propagated by surface tractions P_i distributed uniformly over a distance a at the rear of the crack front and the work done by these surface tractions is equated with the work done in creating new surface of surface energy γ. The medium is of general elastic anisotropy and part of the analysis is a generalization of the static cases considered by Eshelby et al. (4) and Stroh (5).

3. ANALYSIS OF MOTION

The stresses σ_{ij} are related to the elastic displacements u_k by the equations

$$(1) \qquad \sigma_{ij} = C_{ijkl} \frac{\partial u_k}{\partial x_l}$$

where i,j,k,l = 1,2,3 and the convention of summing over a repeated latin suffix is used.

We also have the symmetry relations

$$(2) \qquad C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}$$

where the C_{ijkl} are the elastic constants.

The equations of motion are

$$(3) \qquad \frac{\partial \sigma_{ij}}{\partial x_j} = \rho \frac{\partial^2 u_i}{\partial t^2}$$

where ρ is the density.

Substituting (1) in (3) gives

$$(4) \qquad C_{ijkl} \frac{\partial^2 u_k}{\partial x_l \partial x_j} = \rho \frac{\partial^2 u_i}{\partial t^2} .$$

The C_{ijkl} as given above are the elastic constants of the material referred to the axes (x_1, x_2, x_3). These can be found in terms of the elastic constants referred to the symmetry axes of the crystals by simple formulae for the rotation of axes.

We now assume that u_k is independent of x_3. Also, because we are considering the steady motion of a crack, we make the substitution $X = x_1 - v_1 t$ in (3), and write

$$\sigma_{ij} = \sigma_{ij}(X, x_2)$$

for the stresses. Then

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} = V_1^2 \dfrac{\partial^2}{\partial X^2} \\ \dfrac{\partial^2}{\partial x_1^2} = \dfrac{\partial^2}{\partial X^2} \end{cases}$$

We write

$$u_k = A_k f(X + T x_2)$$

as a solution of (3) provided the constant vector A_k satisfies the equations

$$(5) \qquad (C_{i1k1} - \rho V_1^2 \delta_{ik} + T C_{i1k2} + T C_{i2k1} + T^2 C_{i2k2}) A_k = 0$$

where i = 1,2 or 3 and summation is over k by convention.

Values of $A_k \neq 0$ can be found to satisfy these equations if T is a root of the sextic equation

$$(6) \qquad \left| C_{i1k1} - \rho V_1^2 \delta_{ik} + T(C_{i1k2} + C_{i2k1}) + T^2 C_{i2k2} \right| = 0$$

We can rewrite this as

$$(7) \qquad \left| A - \rho V_1^2 I \right| = 0$$

where A is the matrix

$$\left(C_{i1k1} + T(C_{i1k2} + C_{i2k1}) + T^2 C_{i2k2} \right)$$

and I the unit matrix.

Referring to (4), we start with the tacit assumption that only imaginary roots of (6) are applicable and expect to verify this later.

Under the above restriction, (6) will give us three roots (with positive imaginary part) $T_\alpha (\alpha = 1,2,3)$ with complex conjugates $\overline{T_\alpha}$, the corresponding values of A_k found from (5) we will call (following Stroh) $A_{k\alpha}$ and $\overline{A_{k\alpha}}$.

We then write the displacement as

$$(8) \qquad u_k = \sum_\alpha A_{k\alpha} f_\alpha(z_\alpha) + \sum_\alpha \overline{A_{k\alpha}}\, \overline{f_\alpha(z_\alpha)}$$

where $z_\alpha = X + T_\alpha x_2$ and α goes from 1 to 3.

From (1) we write the stresses as

$$\sigma_{ij} = C_{ijk1} \frac{\partial u_k}{\partial X} + C_{ijk2} \frac{\partial u_k}{\partial x_2}$$

Substituting from (8) we have

$$(9) \qquad \sigma_{ij} = \sum_\alpha L_{ij\alpha} f'_\alpha(z_\alpha) + \sum_\alpha \overline{L_{ij\alpha}}\, \overline{f'_\alpha(z_\alpha)}$$

where

$$L_{ij\alpha} = (C_{ijk1} + C_{ijk2} T_\alpha) A_{k\alpha}$$

and dashes denote differentiation with respect to z_α.

4. BOUNDARY CONDITIONS

On the surface $x_2 = 0$, over the cut $-\infty < X < 0$, we are given

$$\sigma_{12} = g_1(X) \tag{10}$$
$$\sigma_{22} = g_2(X) \qquad -\infty < X < 0$$
$$\sigma_{23} = g_3(X)$$

and

$$R\sigma_{12},\ R\sigma_{22},\ R\sigma_{23} \to 0$$

as

$$R = \{X^2 + x_2^2\}^{\frac{1}{2}} \to \infty$$

where $X = x_1 - v_1 t$.

The functions g_1, g_2 and g_3 are assumed to be differentiable any number of times except perhaps at a finite number of isolated points.

5. SOLUTION OF THE PROBLEM

We consider the semi-infinite regions $x_2 > 0$, $x_2 < 0$ separately. In $x_2 > 0$, we can write as the expression for u_k which remains finite at infinity,

$$u_K = 2\mathrm{Rl} \sum_\alpha A_{K\alpha} \int_0^\infty F_\alpha^+(\beta) \exp(i\beta z_\alpha)\, d\beta. \tag{11}$$

This is seen to be an expression of the form (8) and hence is a solution where $F_\alpha^+(\beta)$ is an arbitrary function.

The stresses are

$$\sigma_{ij} = 2\mathrm{Rl} \sum_\alpha L_{ij\alpha} \int_0^\infty i\beta F_\alpha^+(\beta) \exp(i\beta z_\alpha)\, d\beta \tag{12}$$

For $x_2 < 0$, we write

$$\sigma_{ij} = 2\mathrm{Rl} \sum_\alpha L_{ij\alpha} \int_0^\infty -i\beta F_\alpha^-(\beta) \exp(-i\beta z_\alpha)\, d\beta \tag{13}$$

and a similar expression for u_k where $F_\alpha^-(\beta)$ is an arbitrary function.

We have now to relate the solution in the two regions. For the part $x_2 = 0$, $X > 0$ the material must be joined together, and the boundary condition on the stress is then that the components σ_{i2} should be continuous.

The region $X < 0$, $x_2 = 0$ is already specified by the boundary conditions. Equating (12) and (13) when $x_2 = 0$, we have

$$\sum_\alpha L_{i2\alpha} F_\alpha^+(\beta) = \sum_\alpha \overline{L_{i2\alpha}}\, \overline{F_\alpha^-(\beta)} = \psi_{i2}(\beta), \quad \text{say.} \tag{14}$$

Solving the equations (14), we obtain

$$\begin{cases} F_\alpha^+(\beta) = M_{\alpha i2}\, \psi_{i2}(\beta) \\ F_\alpha^-(\beta) = M_{\alpha i2}\, \overline{\psi_{i2}(\beta)} \end{cases} \tag{15}$$

provided the $L_{i2\alpha}$ are independent.

Substituting in equation (11) we have for $x_2 > 0$

$$u_K = 2\mathrm{Rl} \sum_\alpha A_{K\alpha} M_{\alpha i2} \int_0^\infty \psi_{i2}(\beta) \exp(i\beta z_\alpha)\, d\beta \tag{16}$$

and for $x_2 < 0$

$$u_K = 2\mathrm{Rl} \sum_\alpha A_{K\alpha} M_{\alpha i2} \int_0^\infty \overline{\psi_{i2}(\beta)} \exp(-i\beta z_\alpha)\, d\beta. \tag{17}$$

Subtracting (17) and (16) we find that the difference in displacement on either side of the plane $x = 0$ is

$$\Delta u_K = -2iB_{Ki2} \int_0^\infty \{\psi_{i2}(\beta)\exp(i\beta X) - \overline{\psi_{i2}(\beta)}\exp(-i\beta X)\}\, d\beta \tag{18}$$

where

$$B_{Ki2} = \frac{1}{2} i \sum_\alpha (A_{K\alpha} M_{\alpha i2} - \overline{A_{K\alpha}}\, \overline{M_{\alpha i2}}). \tag{19}$$

Outside the crack the displacement must be zero so for $X > 0$ we have since $|B_{Ki2}| \neq 0$

$$\int_0^\infty \{\psi_{i2}(\beta)\exp(i\beta X) - \overline{\psi_{i2}(\beta)}\exp(-i\beta X)\}\, d\beta = 0 \tag{20}$$

Substituting for (15) in (12) we find the stresses in the region $x_2 > 0$ to be

$$\sigma_{K2} = 2\mathrm{Rl} \sum_\alpha L_{K2\alpha} M_{\alpha i2} \int_0^\infty i\psi_{i2}(\beta)\, \beta \exp(i\beta z_\alpha)\, d\beta. \tag{21}$$

On the surface of the crack for $X < 0$ the stresses are given by (10) which we call $-\Gamma_k(X)$ for brevity.

Then from (21) we can write for $-\infty < X < 0$, $x_2 = 0$

$$-\Gamma_K(X) = 2\mathrm{Rl}\, i \int_0^\infty C_{Ki}\, \psi_{i2}(\beta)\, \beta \exp(i\beta X)\, d\beta \tag{22}$$

where $C_{ki} = \sum_\alpha L_{k2\alpha} M_{\alpha i2}$ is independent of β.

From (14) and (15) we have

$$\sum_\alpha L_{i2\alpha} M_{\alpha j2} = \delta_{ij}$$

Thus

$$C_{Ki} = \delta_{Ki}$$

and (22) becomes

$$-\Gamma_K(X) = i \int_0^\infty \beta\, [\psi_{K2}(\beta)\exp(i\beta X) - \overline{\psi_{K2}(\beta)}\exp(-i\beta X)]\, d\beta \tag{23}$$
$$-\infty < X < 0$$

If in the above we had made the assumption that the crack opened symmetrically we could have solved the problem by a straightforward generalization of [2]. However, it seemed wisest not to make such an assumption even though it turns out that we obtain the same result.

We thus have to solve the dual integral equations (20) and (23) which we do in Appendix 1.

Furthermore, in our subsequent analysis we specialize to the particular problem considered in section B of Appendix 1, i.e. we take

$$\Gamma_K(X) = P_K \quad -a < X < 0; \qquad = 0 \quad -\infty < X < -a, \tag{10}$$

where P_k is a constant.

6. EVALUATION OF THE STRESSES AND RATES OF CHANGE OF DISPLACEMENT

It is now possible to substitute equation (xvi) of Appendix 1 into equations (16), (17) and (21) to calculate the displacements and stresses. However, our main purpose here is to calculate the energy associated with the moving crack so we calculate $\frac{\partial u_K}{\partial t}$.

From (16) for $x_2 > 0$

$$\frac{\partial u_K}{\partial t} = -V_1 \frac{\partial u_K}{\partial X} = -2Rl \sum_\alpha i V_1 A_{K\alpha} M_{\alpha i 2} \int_0^\infty \beta \Psi_{i2}(\beta) \exp(i\beta z_\alpha)\, d\beta \tag{24}$$

The integral

$$\int_0^\infty \beta \Psi_{j2}(\beta) \exp(i\beta z_\alpha)\, d\beta = -i \frac{P_j e^{-i\pi/4}}{\pi^{3/2}} \left\{ \int_0^\infty a^{\frac{1}{2}} \exp(i\beta z_\alpha) \cdot \beta^{-\frac{1}{2}} d\beta - \int_0^{a^{\frac{1}{2}}} \int_0^\infty \frac{e^{i\beta(a-u^2+z_\alpha)}}{\beta^{\frac{1}{2}}} d\beta\, du \right\} \tag{25}$$

where $z_\alpha = X + T_\alpha x_2$.

The imaginary part of T being positive (see section 3), then

$$\int_0^\infty a^{\frac{1}{2}} \beta^{-\frac{1}{2}} \exp(i\beta z_\alpha)\, d\beta = a^{\frac{1}{2}} e^{+i\pi/4} \pi^{\frac{1}{2}} z_\alpha^{-\frac{1}{2}}$$

and

$$\int_0^\infty \frac{e^{i\beta(a-u^2+z_\alpha)}}{\beta^{\frac{1}{2}}} d\beta = e^{+i\pi/4} \pi^{\frac{1}{2}} (a - u^2 + z_\alpha)^{-\frac{1}{2}}$$

Thus

$$\int_0^{a^{\frac{1}{2}}} \int_0^\infty \frac{e^{i\beta(a-u^2+z_\alpha)}}{\beta^{\frac{1}{2}}} d\beta\, du = e^{+i\pi/4} \pi^{\frac{1}{2}} \int_0^{a^{\frac{1}{2}}} \frac{du}{(a-u^2+z_\alpha)^{\frac{1}{2}}} = \frac{e^{+i\pi/4}\pi^{\frac{1}{2}}}{i} \left[\log \frac{a^{\frac{1}{2}} + i z_\alpha^{\frac{1}{2}}}{(a+z_\alpha)^{\frac{1}{2}}} - \frac{i\pi}{2} \right] \tag{26}$$

Thus we can write

$$\frac{\partial u_K}{\partial t} = -Rl \sum_\alpha \frac{i V_1 A_{K\alpha} M_{\alpha i 2} P_i}{\pi} \left[\log\left(\frac{a^{\frac{1}{2}} - i z_\alpha^{\frac{1}{2}}}{a^{\frac{1}{2}} + i z_\alpha^{\frac{1}{2}}} \right) + i\pi - \frac{2ia^{\frac{1}{2}}}{z_\alpha^{\frac{1}{2}}} \right] \tag{27}$$

for $x_2 > 0$.

We obtain the same result for $x_2 < 0$, so we can infer that the crack opens symmetrically.

When $X < 0$ and $x_2 = 0$, we write $\tau = -X$ then (27) becomes

$$\frac{\partial u_K}{\partial t} = \frac{-1}{\pi} \left\{ D_K \left[\log\left(\frac{a^{\frac{1}{2}} - \tau^{\frac{1}{2}}}{a^{\frac{1}{2}} + \tau^{\frac{1}{2}}} \right) - \frac{2a^{\frac{1}{2}}}{\tau^{\frac{1}{2}}} \right] + E_K \pi \right\} \tag{28}$$

where

$$D_K = i V_1 \sum_\alpha P_i (A_{K\alpha} M_{\alpha i 2} - \overline{A_{K\alpha}}\, \overline{M_{\alpha i 2}}) \tag{28a}$$

$$E_K = -V_1 \sum_\alpha P_i (A_{K\alpha} M_{\alpha i 2} + \overline{A_{K\alpha}}\, \overline{M_{\alpha i 2}}).$$

7. THE CALCULATION OF THE ENERGY

The rate at which the external forces do work is

$$W = \int_{-a}^0 P_K \left[\frac{\partial u_K}{\partial t} \right]_{x_2 = 0} dX \tag{29}$$

Replacing $-X$ by τ and substituting (28) in (29), we find after integrating, that

$$W = \frac{-1}{\pi} \left\{ \pi a E_K P_K - 2a D_K P_K \right\}. \tag{30}$$

Using the energy criterion of Griffith [6] in the same way as Craggs [1], we equate the rate of loss of mechanical energy at the point $X = x_2 = 0$ to the rate of increase of surface energy of the material, which is $2v_1\gamma$ where γ is the energy per unit area of the surface, giving the resulting equation

$$2V_1\gamma = a \left\{ \frac{2}{\pi} D_K P_K - E_K P_K \right\} \tag{31}$$

8. APPLICATION TO CRYSTALS OF CUBIC SYMMETRY

In the remainder of the paper we apply the preceeding general theory to the case of a crack under internal pressure propagating through a medium of cubic symmetry on an arbitrary fracture plane. The elastic properties of a cubic crystal can be specified with respect to the cube axes by three independent constants. The three which are usually tabulated [7] are either (c_{11}, c_{12}, c_{44}) or (s_{11}, s_{12}, s_{44}) but it is convenient here to use (A, B, c_{44}) where

$$A = \frac{2C_{44}}{C_{11} - C_{12}} \tag{32}$$

is Zener's anisotropy ratio (8) and

(33) $$B = \frac{C_{11} + 2C_{12}}{C_{44}}$$

is the ratio of bulk modulus to shear modulus.

Since we are only considering an internal pressure the components P_1 and P_3 of the surface tractions are zero. It is convenient to specify the pressure P_2 in non dimension form as

(34) $$P = P_2 / (\gamma C_{44}/a)^{\frac{1}{2}}$$

and the non dimensional velocity of the crack as

(35) $$V = V_1 / (C_{44}/\rho)^{\frac{1}{2}}$$

where the denominator is the shear wave velocity along a cube axis. The governing equation (31) can then be written in the non dimensional form

(36) $$P^2 = \frac{\pi}{2} H(V, A, B)$$

where H is a function of the non dimensional quantities V, A and B and also of course the orientation of the crack plane and direction of propagation with respect to the cube axes.

In general, H can only be found numerically but its general form is illustrated in Fig. 1, which is for the specific case of a crack in iron propagating on a $\{100\}$ plane with the crack front along $\langle 100 \rangle$. This is similar to that found by Craggs [1] for an isotropic medium. It will be seen that the pressure necessary to propagate the crack falls as the velocity increases. To interpret this physically we must also consider the regulation of the source of the pressure. If the velocity of the crack increases then the pressure will be supplying energy at a greater rate and in general the pressure will fall, as suggested by the dashed line in Fig. 1. This would give stable propagation at the velocity corresponding to the point of intersection in Fig. 1.

9. CRACKS PRODUCED BY HYDROGEN CHARGING

Gell and Robertson [3] have examined the cracks produced in a Fe-3% Si single crystal by hydrogen introduced into the crystal by electrolytic charging. They observed cracks on the usual $\{100\}$ fracture planes with the unusual feature that these cracks were very nearly square in plan with the crack front running along the $\langle 110 \rangle$ directions in the fracture plane. They deduced that the crack had grown intermittently and that the crack had remained square during several cycles of growth and quiescence. They suggest that the reason for intermittent growth is that the crack starts to spread when the hydrogen pressure in the crack equals the stress necessary for propagation in the Griffith criterion, but the crack soon stops because the hydrogen pressure will fall as the crack grows and further growth must wait for the diffusion of more hydrogen through the iron into the cavity of

the crack. From the square shape of the crack they conclude that the easy directions of crack growth must be $\langle 100 \rangle$ and that $\langle 110 \rangle$ directions are most difficult so that the crack is outlined by the latter.

We suggest that the easy and difficult directions of propagation may be determined by elastic anisotropy and apply the preceeding theory to explain the shape of the cracks observed by Gell and Robertson and predict the shapes which would be observed if the same type of experiment were repeated on other metals. It is obvious that the hydrogen pressure is a driving force which is very poorly regulated and so it would be expected from Fig. 1 that the crack velocity would never be a large fraction of the velocityof sound. From Fig. 1 it will be seen that even for V = 0.3 the necessary pressure is almost equal to that for V = 0 and it is convenient just to consider the pressure necessary for a vanishingly small crack velocity. This pressure is shown in Fig. 2 as a function of the orientation of the direction of the crack front in the $\{100\}$ plane for both iron and molybdenum. Only the variation over 45° is shown inFig. 2 since due to the four-fold symmetry in the $\{100\}$ plane, the other half quadrants would show the appropriate repetition of Fig. 2. This is the non dimensional pressure and the surface energy γ in (34) will be the surface energy of the $\{100\}$ plane. It will be seen that for iron the $\langle 110 \rangle$ direction is indeed the direction which needs the greatest pressure to start propagation but that for molybdenum the situation is reversed so that similar cracks in molybdenum would be predicted to be squares bounded by $\langle 100 \rangle$ directions as indicated in Fig. 3.

For two directions in the $\{100\}$ plane an explicit expression follows from the analysis of Atkinson (2). These are for the crack front along $\langle 100 \rangle$

(37) $$H(V=0, A, B) = 2(1+AB) / \left\{ A(4+AB)(AB+3A+1) \right\}^{\frac{1}{2}}$$

and for the crack front along $\langle 110 \rangle$

(38) $$H(V=0, A, B) = \left\{ \frac{3A(AB+3B+4)(Q+AB-2)}{(AB+3A+1)(Q+AB+6A-2)} \right\}^{\frac{1}{2}}$$

where

(39) $$Q = \left\{ (AB+4)(AB+3A+1) \right\}^{\frac{1}{2}}$$

and the corresponding pressures follow from (36).

In contrast to iron and molybdenum which have a $\{100\}$ fracture plane, vanadium and tantalum fracture on a $\{110\}$ plane. For these two metals Fig. 4 shows the pressure (for zero crack velocity) as a function of the orientation of the crack front in a $\{110\}$ plane. To convert these to true pressures, the surface energy of the $\{110\}$ plane must be used in (34). From this we can predict that cracks formed by hydrogen in vanadium should be bounded as far as possible by $\langle 100 \rangle$ directions and for tantalum by $\langle 110 \rangle$ directions. This leads to the conclusion, shown schematically in Fig. 5, that the cracks in vanadium on a $\{110\}$ plane should be elongated in a $\langle 100 \rangle$ direction and cracks in tantalum elongated in a $\langle 110 \rangle$ direction.

As for the $\{100\}$ plane, there are explicit expressions for cracks on the

$\{110\}$ plane. For a crack front along $\langle 100 \rangle$, H is the same as given in (37). For a crack along $\langle 110 \rangle$

$$H(V=0, A, B) = \left\{ \frac{3A(AB+3B+4)(Q+AB-2)}{(AB+4)(Q+AB+6A-2)} \right\}^{\frac{1}{2}} \tag{40}$$

with Q given by (39). In addition there is the $\langle 111 \rangle$ direction in this plane. Although this is not a direction which is perpendicular to a symmetry plane and therefore is not covered by Atkinson (2), it is a direction of three-fold symmetry and a small extension of the analysis of a dislocation in a $\langle 111 \rangle$ direction by Head (9) gives for the crack in the $\langle 111 \rangle$ direction

$$H(V=0, A, B) = \left\{ \frac{4(2A+1)(B+1)^2}{(AB+A+2B+1)(AB+3A+1)} \right\}^{\frac{1}{2}} \tag{41}$$

From these explicit results for the three directions of high symmetry of a cubic crystal, some idea may be gained of the effect of elastic anisotropy on crack propagation in other cubic materials for which elastic constants are known. However it would seem wisest to extend this (by numerical methods) to a coverage of a range of orientations of the crack front so as to reveal any latent peculiarities. One example of this can be seen in Fig. 4 for vanadium for which the most favoured direction of propagation is non crystallographic at $\sim 70^{\circ}$ from $\langle 100 \rangle$. It is true that the symmetry of the crystal requires, on both the $\{100\}$ and $\{110\}$ planes, that the $\langle 100 \rangle$ and $\langle 110 \rangle$ directions are extremes but it is not necessary that these are the only extremes, nor that one must be a maximum and the other a minimum.

Electrolytic hydrogen charging as used by Gell and Robertson appears to be a very useful tool in the study of the fundamentals of fracture. It has some unique fractures which would not be easy to duplicate. The fracture is internal so that the fracture surface should remain clean, being in contact only with hydrogen gas. The stress system is a simple internal gas pressure although the actual pressure is not easily measured. Finally the poor regulation characteristic of the gas pressure which gives the cyclic growth and quiescence of the crack also gives a natural amplification of any differences which may exist in the ease of crack propagation in different directions.

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APPENDIX I

A. SOLUTION OF THE INTEGRAL EQUATIONS

The equations to solve are

$$\text{(i)} \quad \int_0^{\infty} \{ \Psi_{i2}(\beta)\exp(i\beta X) - \overline{\Psi_{i2}(\beta)}\exp(-i\beta X) \} d\beta = 0 \quad \text{for } X > 0$$

$$\text{(ii)} \quad i\int_0^{\infty} \beta \{ \Psi_{i2}(\beta)\exp(i\beta X) - \overline{\Psi_{i2}(\beta)}\exp(-i\beta X) \} d\beta = -\Gamma_i(X) \quad \text{for } X < 0.$$

Differentiating (i) with respect to X gives

$$\text{(iii)} \quad i\int_0^{\infty} \beta \{ \Psi_{i2}(\beta)\exp(i\beta X) + \overline{\Psi_{i2}(\beta)}\exp(-i\beta X) \} d\beta = 0 \quad \text{for } X > 0.$$

In (ii) replace X by $(X - \varsigma)$, multiply through by an arbitrary function $N_2(\varsigma)$ and integrate over ς from 0 to ∞. Similarly in (iii) replace X by $X + \varsigma$, multiply by $N_1(\varsigma)$ and integrate over ς from 0 to ∞. (ii) and (iii) become

$$\text{(iv)} \quad i\int_0^{\infty} \beta \{ \Psi_{i2}(\beta)\exp(i\beta X) M_2(\beta) - \overline{\Psi_{i2}(\beta)}\exp(-i\beta X) L_2(\beta) \} d\beta = -\int_0^{\infty} \Gamma_i(X-\varsigma) N_2(\varsigma) d\varsigma \quad \text{for } X < 0$$

and

$$\text{(v)} \quad i\int_0^{\infty} \beta \{ \Psi_{i2}(\beta)\exp(i\beta X) M_1(\beta) + \overline{\Psi_{i2}(\beta)}\exp(-i\beta X) L_1(\beta) \} d\beta = 0 \quad \text{for } X > 0$$

respectively, where

$$\text{(vi)} \quad \left\{ \begin{aligned} M_2(\beta) &= \int_0^{\infty} N_2(\varsigma) e^{-i\beta\varsigma} d\varsigma \\ L_2(\beta) &= \int_0^{\infty} N_2(\varsigma) e^{+i\beta\varsigma} d\varsigma \\ M_1(\beta) &= \int_0^{\infty} N_1(\varsigma) e^{+i\beta\varsigma} d\varsigma \\ L_1(\beta) &= \int_0^{\infty} N_1(\varsigma) e^{-i\beta\varsigma} d\varsigma \end{aligned} \right.$$

If we now choose N and N so that

$$\text{(vii)} \quad \left\{ \begin{aligned} M_2(\beta) &= M_1(\beta) K \\ -L_2(\beta) &= L_1(\beta) K \end{aligned} \right.$$

where K is a constant, we can rewrite (iv) and (v) as

(viii) $$i\int_0^\infty \beta\left\{\psi_{i2}(\beta)\exp(i\beta X)M_1(\beta) + \overline{\psi_{i2}(\beta)}\exp(-i\beta X)L_1(\beta)\right\}d\beta$$

$$= \begin{cases} 0 & X>0 \\ \dfrac{-1}{K}\displaystyle\int_0^\infty \Gamma_i(X-\xi)N_2(\xi)\,d\xi & X<0 \end{cases}$$

We have the well known results

(ix) $$\begin{cases} \displaystyle\int_0^\infty \xi^{-\frac{1}{2}}e^{\pm i\xi\beta}d\xi = e^{\pm i\pi/4}\pi^{\frac{1}{2}}\beta^{-\frac{1}{2}} & \beta>0 \\ \displaystyle\int_{-\infty}^\infty \xi^{-\frac{1}{2}}e^{\pm i\xi\beta}d\xi = 2e^{\pm i\pi/4}\pi^{\frac{1}{2}}\beta^{-\frac{1}{2}} & \beta>0 \\ \text{so that} & \\ \displaystyle\int_{-\infty}^0 \xi^{-\frac{1}{2}}e^{\pm i\xi\beta}d\xi = e^{\pm i\pi/4}\pi^{\frac{1}{2}}\beta^{-\frac{1}{2}} & \beta>0. \end{cases}$$

Thus choosing

$$N_1(\xi) = \xi^{-\frac{1}{2}}, \quad N_2(\xi) = \xi^{-\frac{1}{2}}$$

we have

(x) $$\begin{cases} M_2(\beta) = e^{-i\pi/4}\pi^{\frac{1}{2}}\beta^{-\frac{1}{2}}; & M_1(\beta) = e^{+i\pi/4}\pi^{\frac{1}{2}}\beta^{-\frac{1}{2}} \\ L_2(\beta) = e^{+i\pi/4}\pi^{\frac{1}{2}}\beta^{-\frac{1}{2}}; & L_1(\beta) = e^{-i\pi/4}\pi^{\frac{1}{2}}\beta^{-\frac{1}{2}}. \end{cases}$$

Equations (vii) are satisfied if we take k = −i so that

$$L_1(\beta) = \overline{M_1(\beta)}.$$

Then equation (viii) becomes

(xi) $$\int_0^\infty \beta\left\{\chi_{K2}(\beta)\exp(i\beta X) + \overline{\chi_{K2}(\beta)}\exp(-i\beta X)\right\}d\beta$$

$$= \begin{cases} 0 & X>0 \\ -\displaystyle\int_0^\infty \Gamma_K(X-\xi)\xi^{-\frac{1}{2}}d\xi & X<0 \end{cases}$$

where

$$\chi_{K2}(\beta) = \psi_{K2}(\beta)M_1(\beta).$$

Writing

$$\chi_{K2}(\beta) = \chi'_K + i\chi''_K$$

where χ'_K, χ''_K are real functions, the left hand side of (xi) becomes

$$\int_0^\infty 2\beta\left\{\chi'_i(\beta)\cos\beta X - \chi''_i(\beta)\sin\beta X\right\}d\beta$$

thus we can find χ'_i and χ''_i by the Fourier inversion theorem.

B. A PARTICULAR PROBLEM

Take
$$\Gamma_K(X) = P_K \qquad -a<X<0$$
$$= 0 \qquad -\infty<X<-a$$

where P_K is a constant.

Then
$$\int_0^\infty \Gamma_K(X-\xi)N_2(\xi)\,d\xi = \begin{cases} 0 & -\infty<X<-a \\ 2P_K(X+a)^{\frac{1}{2}} & -a<X<0 \end{cases}$$

so applying the Fourier inversion theorem to equation (xi) we find

(xii) $$\begin{cases} \beta\chi'_K(\beta) = \dfrac{-1}{\pi}\displaystyle\int_{-a}^0 P_K(X+a)^{\frac{1}{2}}\cos\beta X\,dX \\ \beta\chi''_K(\beta) = \dfrac{+1}{\pi}\displaystyle\int_{-a}^0 P_K(X+a)^{\frac{1}{2}}\sin\beta X\,dX. \end{cases}$$

After a simple change of variable these integrals can be seen to be the standard Fresnel integrals, the solution of which is given as an infinite converging series; however as we are primarily concerned with calculating the rates of change of displacement in order to calculate the energy associated with the moving crack we can avoid them by a change in the order of integration.

We can rewrite equations (xii) as

(xiii) $$\begin{cases} \beta\chi'_K(\beta) = \dfrac{-1}{\pi}\left(\dfrac{IP_K\sin\beta a}{\beta^{3/2}} - \dfrac{JP_K\cos\beta a}{\beta^{3/2}}\right) \\ \beta\chi''_K(\beta) = \dfrac{1}{\pi}\left(\dfrac{a^{\frac{1}{2}}P_K}{\beta} - \dfrac{JP_K\sin\beta a}{\beta^{3/2}} - \dfrac{IP_K\cos\beta a}{\beta^{3/2}}\right) \end{cases}$$

where

(xiv) $$\begin{cases} I = \displaystyle\int_0^{\beta^{\frac{1}{2}}a^{\frac{1}{2}}}\cos v^2\,dv \\ J = \displaystyle\int_0^{\beta^{\frac{1}{2}}a^{\frac{1}{2}}}\sin v^2\,dv \\ I+iJ = \displaystyle\int_0^{\beta^{\frac{1}{2}}a^{\frac{1}{2}}}e^{iv^2}dv \\ I-iJ = \displaystyle\int_0^{\beta^{\frac{1}{2}}a^{\frac{1}{2}}}e^{-iv^2}dv = \int_0^{a^{\frac{1}{2}}}e^{-i\beta u^2}\beta^{\frac{1}{2}}du \end{cases}$$

are the Fresnel integrals, and equations (xiii) are obtained from (xii) simply by integrating by parts and changing the variable of integration.

Also
$$\chi_{K2}(\beta) = \chi'_K(\beta) + i\chi''_K(\beta)$$

Thus
$$= \psi_{K2}(\beta)M_1(\beta).$$

(xv) $$\psi_{K2}(\beta) = \frac{e^{-i\pi/4}}{\pi^{\frac{1}{2}}}\beta^{\frac{1}{2}}\chi_{K2}(\beta).$$

From (xiii) and (xv) we have

(xvi) $$\psi_{K2}(\beta) = \frac{-i\,e^{-i\pi/4}\beta^{-\frac{1}{2}}}{\pi\cdot\pi^{\frac{1}{2}}}P_K\left\{\frac{a^{\frac{1}{2}}}{\beta} - \frac{e^{ia}}{\beta^{3/2}}(I-iJ)\right\}$$

which is the solution of the integral equations given in terms of Fresnel integrals.

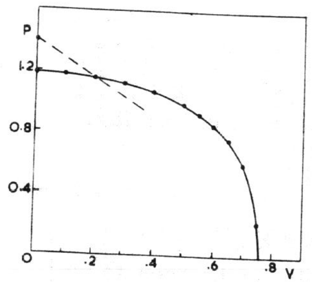

Fig. 1 Variation of pressure with velocity for a crack in iron on $\{100\}$ plane, $\langle 100 \rangle$ crack front.

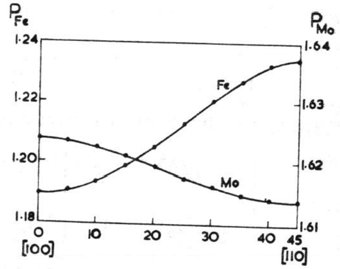

Fig. 2 Variation of pressure with orientation in a $\{100\}$ plane for iron and molybdenum.

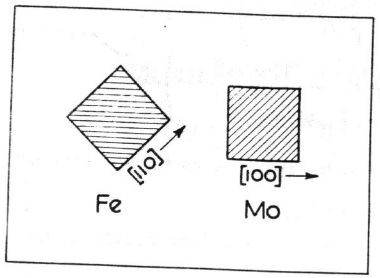

Fig. 3 Plan of crack shape in iron and molybdenum.

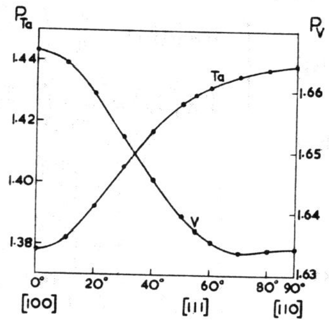

Fig. 4 Variation of pressure with orientation in a $\{110\}$ plane for vanadium and tantalum.

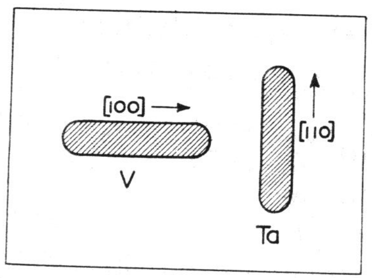

Fig. 5 Plan of crack shape in vanadium and tantalum.