A-6 STATIONARY CRACK AND CONTINUOUS DISTRIBUTIONS OF DISLOCATIONS*

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ABSTRACT

Two isolated crack problems are studied. The first is a slit under anti-plane shear and the second is a penny-shaped crack. A Tresca yield condition is used in both cases to ensure that all stresses are bounded. A continuous distribution of dislocations is computed for these problems and curves are developed that show the length of the plastic zone and the displacement at the tip of the crack as a function of applied stress.

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1. INTRODUCTION

Solutions to many two- and three-dimensional crack problems in classical elasticity theory are well-known and are generally found by means of techniques developed for axially symmetric potential problems. A very extensive summary of such solutions is given by Sneddon [1].* Recent work such as that of Collins [2] has considered the more difficult asymmetric crack problems.

The solutions that are derived from potential theory alone have a fundamental difficulty, that is, although the displacements are finite at the edge of the crack, the stress there becomes infinite. This is a clear violation of the requirement in elasticity that the stresses remain bounded. One method of coping with this problem is that of Barenblatt [3]. He postulates a system of stresses existing at the edge of the crack that inhibit the occurrence of infinite stresses there. These he calls "cohesive stresses" and he develops a universal constant called the Modulus of Cohesion to define the stresses. He specifies these stresses as a local phenomenon occurring just at the tip of the crack, and their magnitude is just sufficient to cause the usual singularity to vanish. This method is largely restricted to the class of brittle materials.

Dugdale [4] proposed another method for removing this singularity. The requirement is made that near the edge of the crack a plastic yield condition is satisfied. The region over which yield occurs is determined by requiring that the stresses remain finite everywhere. Solution to another such problem is given by Hult and McClintock [5]. The approach of these authors is based upon methods appropriate to the elasto-plasticity. The condition of plasticity is satisfied on the plane normal to the crack. The problems studied here are two-dimensional or anti-plane problems.

Bilby, Cottrell and Swinden [6] have considered the same problem from the standpoint of the theory of distributions of dislocations. In this paper a calculation is made of the length of the plastic zone needed to accommodate a given plastic displacement at the root of a notch in a uniformly stressed solid. A later paper by Bilby, Cottrell, Smith and Swinden [7] considers the behavior of an infinite array of cracks subject to a uniform stress at infinity. The purpose of this work is to show the relation between the results of England [8], England and Green [9] and Collins [2] and those of Bilby, et al. [6,7]. It will be shown that one can transform the solution of a crack problem with the required plasticity condition to obtain the distribution of dislocations for the anti-plane and the penny-shaped crack.

2. ANTI-PLANE STRAIN PROBLEM

The equivalency of the two approaches, the method of continuous

*Numbers in brackets refer to references listed at the end of this paper.

distribution of dislocations and Dugdale's application of classical theory of elasticity, is shown by the anti-plane strain problem (Figure 1). For this problem the boundary conditions are the following:

$$\begin{aligned} \tau_{yz} &= R & y &= \infty & & \\ \tau_{yz} &= 0 & y &= 0 & & (|x| < c) \\ \tau_{yz} &= k & y &= 0 & & (c < |x| < a) \\ w &= 0 & y &= 0 & & (a < |x|) \end{aligned} \tag{1}$$

where R is the shear stress at ∞, and k is the yielding shear stress of the material, and w is the displacement in the z direction. The problem can be reduced to

$$\begin{aligned} \tau_{yz} &= 0 & y &= \infty & & \\ \tau_{yz} &= -R & y &= 0 & & (|x| < c) \\ \tau_{yz} &= k - R & y &= 0 & & (c < |x| < a) \\ w &= 0 & y &= 0 & & (a < |x|) \end{aligned} \tag{2}$$

by subtracting the uniform stress R. The above problem can be solved easily by use of the potentials introduced by England and Green [9] and later used by England [8] for general solutions to anti-plane strain problems. The solution is

$$\begin{aligned} &\sigma_x = \sigma_y = \sigma_z = \sigma_{xy} = 0 \\ &\tau_{xz} + i\,\tau_{yz} = \mu\, \bar{\Omega}'(\bar{z}) \\ &2w = \Omega(z) + \bar{\Omega}(\bar{z}) \end{aligned} \tag{3}$$

where

$$\begin{aligned} \Omega(z) &= i \int_0^a \frac{F(t)\,dt}{(z^2-t^2)^{\frac{1}{2}}} \\ F(t) &= \frac{2t}{\pi} \int_0^t \frac{f(x)\,dx}{(t^2-x^2)^{\frac{1}{2}}} \\ \mu\, f(x) &= -(\tau_{yz})_{y=0} \qquad (|x| < a) \end{aligned} \tag{4}$$

The quantity which is particularly important is w at y = ± 0 and |x| < a. It follows from equations (3) and (4) that

$$w = \pm \int_{|x|}^{a} \frac{F(t)\,dt}{(t^2-x^2)^{\frac{1}{2}}} \qquad y = \pm 0 (|x| < a) \tag{5}$$

From the given boundary conditions (2) and equation (4), we have

$$F(t) = t\, R/\mu\ , \qquad (t < c)$$

$$F(t) = \frac{2t}{\pi\mu}\left[R \sin^{-1}(c/t) + (R-k)\cos^{-1}(c/t)\right]\ , \qquad (c \le t \le a) \tag{6}$$

There will in general be a stress singularity at $x = a$. To remove this singularity we put

$$F(a) = 0 \tag{7}$$

which leads to the following relation between the applied stress and the spread of the plastic region:

$$\frac{\pi}{2} R/k = \cos^{-1} c/a \tag{8}$$

Equation (8) agrees with the result of Bilby, Cottrell and Swinden [6] who approached the same problem by the continuous distribution of dislocations. In our method the distribution of dislocations, α_{33}, which is the line density of dislocations lying along the z direction with the Burgers vector in the z direction is defined as

$$-\alpha_{33} = (w')_{y=0^+} - (w')_{y=0^-} \tag{9}$$

since the plastic displacement is defined as

$$\int_x^a \alpha_{33}\, dx = (w)_{y=0^+} - (w)_{y=0^-} \tag{10}$$

Substituting (6) into (5) leads to

$$(w)_{y=0^+} = -(w)_{y=0^-} = \frac{k}{\pi\mu}\left[(x+c)\cosh^{-1}\left(\left|\frac{m}{c+x} + n\right|\right) - (x-c)\cosh^{-1}\left(\left|\frac{m}{c-x} + n\right|\right)\right] \tag{11}$$

where

$$m = (a^2 - c^2)/a$$

$$n = c/a\ .$$

The above result agrees also with Bilby, Cottrell and Swinden's result by use of (10). According to Bilby, Cottrell and Swinden, the equation (8) defines the plastic domain, a/c, as the function of the applied stress R. The critical plastic displacement at the root of the crack sufficient for fracture is determined from

$$\frac{\Phi\,\mu}{k\,a} = \frac{4}{\pi}\frac{c}{a}\cosh^{-1}\left[\frac{1}{2}\left(\frac{c}{a} + \frac{a}{c}\right)\right] \tag{12}$$

where

$$\int_c^a \alpha_{33}\, dx = \Phi$$

and $\Phi_{crit.}$ is chosen as order of $10^{-3}a$. The smallest applied stress which gives the value of c/a satisfying (12) is defined as the critical stress for fracture. It should be emphasized here that our theory is completely equivalent to Bilby, Cottrell and Swinden's theory.

3. PENNY-SHAPED CRACK

It is easy to extend the theory to the three-dimensional case. Under an applied tension T in the z direction, a plastic domain is accumulated in the region, $c \le r \le a$, of a penny-shaped crack which has a radius c (Figure 2). The crack is defined by having its center and axis coincide with the center and z axis of a cylindrical coordinate system (r,θ,z). Dugdale's hypothesis requires the condition

$$\begin{aligned}
&\sigma_z = T && z = \infty && \\
&\sigma_z = \sigma_{zr} = \sigma_{z\theta} = 0 && z = 0 && (0 < r < c)\\
&\tfrac{1}{2}(\sigma_z - \sigma_\theta) = k && z = 0 && (c < r < a)\\
&\sigma_{zr} = \sigma_{z\theta} = w = 0 && z = 0 && (a < r)
\end{aligned} \tag{13}$$

The above problem reduces to the following:

$$\begin{aligned}
&\sigma_z = 0 && z = \infty && \\
&\sigma_z = -T && z = 0 && (0 < r < c)\\
&\tfrac{1}{2}(\sigma_z - \sigma_\theta) = k - T/2 && z = 0 && (c < r < a)\\
&w = 0 && z = 0 && (a < r)\\
&\sigma_{zr} = \sigma_{z\theta} = 0 && z = 0 &&
\end{aligned} \tag{14}$$

The justification that $\frac{1}{2}(\sigma_z - \sigma_\theta)$ gives the maximum stress in Tresca's yield condition will be proved after the solution has been obtained. The above problem (14) can be solved by the method employed by Collins (1962). The solution is expressed by a potential function φ,

$$\sigma_z = -\frac{\partial^2\varphi}{\partial z^2} + z\frac{\partial^3\varphi}{\partial z^3}$$

$$\sigma_\theta = -\frac{\partial^2\varphi}{\partial z^2} - (1-2\nu)\frac{\partial^2\varphi}{\partial r^2} + \frac{z}{r}\frac{\partial^2\varphi}{\partial r\partial z}$$

$$\sigma_{zr} = z\frac{\partial^3\varphi}{\partial r\partial z^2}$$

$$\sigma_{z\theta} = \frac{z}{r}\frac{\partial^3\varphi}{\partial\theta\partial z^2}$$

$$\sigma_r = \frac{\partial^2\varphi}{\partial r^2} + \frac{2\nu}{r}\frac{\partial\varphi}{\partial r} + z\frac{\partial^3\varphi}{\partial r^2\partial z}$$

$$2\mu w = -2(1-\nu)\frac{\partial\varphi}{\partial z} + z\frac{\partial^2\varphi}{\partial z^2} \tag{15}$$

where

$$\frac{\partial\varphi}{\partial z} = \frac{1}{2i}\int_{-a}^{a} f(t)[r^2+(z+it)^2]^{-\frac{1}{2}}\,dt$$

$$\frac{\partial\varphi}{\partial r} = \frac{r}{2i}\int_{-a}^{a} f(t)\left\{z+it+[r^2+(z+it)^2]^{\frac{1}{2}}\right\}^{-1}[r^2+(z+it)^2]^{-\frac{1}{2}}\,dt$$

$$f(t) = -f(-t) \tag{16}$$

The unknown function $f(t)$ in (16) can be determined by the second and third condition in (14) by using (15) and (16). Since we have

$$\lim_{z\to 0}[r^2+(z+it)^2]^{\frac{1}{2}} = \lim_{z\to 0}[r^2+(z-it)^2]^{\frac{1}{2}} = (r^2-t^2)^{\frac{1}{2}} \qquad (r > t) \tag{17}$$

and

$$\lim_{z\to 0}[r^2+(z+it)^2]^{\frac{1}{2}} = -\lim_{z\to 0}[r^2+(z-it)^2]^{\frac{1}{2}} = i(t^2-r^2)^{\frac{1}{2}} \qquad (t > r) \tag{18}$$

the derivatives of φ as $z \to 0$ become

$$\left(\frac{\partial\varphi}{\partial z}\right)_{z=0} = -\int_r^a \frac{f(t)dt}{(t^2-r^2)^{\frac{1}{2}}} \qquad (0 \le r \le a)$$

$$= 0 \qquad (a < r < \infty)$$

$$\left(\frac{\partial^2\varphi}{\partial z^2}\right)_{z=0} = \frac{1}{r}\frac{d}{dr}\int_0^r \frac{t\,f(t)dt}{(r^2-t^2)^{\frac{1}{2}}} \qquad (0 \le r \le a)$$

$$\left(\frac{\partial\varphi}{\partial r}\right)_{z=0} = -\frac{1}{r}\int_0^r \frac{t\,f(t)dt}{(r^2-t^2)^{\frac{1}{2}}} \qquad (0 \le r \le a) \tag{19}$$

The conditions in (14), therefore, lead to

$$-\frac{1}{r}\frac{d}{dr}\int_0^r \frac{t\,f(t)dt}{(r^2-t^2)^{\frac{1}{2}}} = -T \qquad (r < c)$$

$$-\frac{(1-2\nu)}{2}\frac{d}{dr}\left\{\frac{1}{r}\int_0^r \frac{t\,f(t)dt}{(r^2-t^2)^{\frac{1}{2}}}\right\} = k - \frac{T}{2} \qquad (c < r < a) \tag{20}$$

The function $f(t)$ can be determined from (20) as

$$\frac{\pi}{2}f(t) = T\,t \qquad (0 < t < c)$$

$$\frac{\pi}{2}f(t) = T\,t - \frac{4k-(1+2\nu)T}{1-2\nu}\left\{(t^2-c^2)^{\frac{1}{2}} - \frac{c}{2}\sin^{-1}\frac{c}{t}\right\} \qquad (c < t < a) \tag{21}$$

It is now seen that $\frac{1}{2}(\sigma_z - \sigma_\theta)$ is the maximum shear stress, since by using (21), (19), and (15), we have at $z = 0$ $(c \le r \le a)$

$$\sigma_z - \sigma_\theta = 2k - T$$

$$\sigma_z - \sigma_r = 2k - T - \left\{2k-(1+2\nu)\frac{T}{2}\right\}\frac{c}{r}$$

$$\sigma_r - \sigma_\theta = \left\{2k-(1+2\nu)\frac{T}{2}\right\}\frac{c}{r}. \tag{22}$$

The relation between the plastic domain and the applied stress can be obtained from the condition

$$f(a) = 0 \tag{23}$$

which is a requirement that the stress singularity vanish at $r = a$. The above equation leads to

$$\frac{(1-2\nu)\left(\frac{T}{2k}\right)}{2-(1+2\nu)\left(\frac{T}{2k}\right)} = \left\{1-\left(\frac{c}{a}\right)^2\right\}^{\frac{1}{2}} - \frac{c}{2a}\cos^{-1}\frac{c}{a} \tag{24}$$

Equation (24) is shown in Figure 3. The similar result (8) for the slit under anti-plane shear and the result for infinite array of slits which was obtained by Bilby, Cottrell, Smith and Swinden [7], are also shown for comparison. It shows that the case of penny-shaped crack needs the largest stress for a given value of a/c. The reason is that the penny-shaped crack receives the largest constraint from the matrix.

The distribution of dislocations which are running in the θ direction with the Burgers vector in the z direction can be defined as

$$-\alpha_{\theta z} = \left(\frac{dw}{dr}\right)_{z=0^+} - \left(\frac{dw}{dr}\right)_{z=0^-} \tag{25}$$

and the plastic displacement as

$$\int_r^a \alpha_{\theta z}dr = (w)_{z=0^+} - (w)_{z=0^-} \tag{26}$$

The displacement w can be obtained easily from (15), (19) and (21) as

$$\frac{\pi}{4}\cdot\frac{2\mu(w)_{z=0^+}}{(1-\nu)} = -\frac{\pi}{4}\cdot\frac{2\mu(w)_{z=0^-}}{(1-\nu)}$$

$$= T(a^2-r^2)^{\frac{1}{2}} - \frac{4k-(1+2\nu)T}{(1-2\nu)}\left[\int_r^a \frac{(t^2-c^2)^{\frac{1}{2}}}{(t^2-r^2)^{\frac{1}{2}}}dt - \right.$$

$$\left. - \frac{c}{2}\int_r^a \cos^{-1} c/t \frac{dt}{(t^2-r^2)^{\frac{1}{2}}}\right] \qquad (c \le r \le a) \tag{27}$$

The plastic displacement at the tip of the crack is of special interest and is given as follows:

$$\int_c^a \alpha_{\theta z}dr = \Phi \tag{28}$$

By using (26) and (27) in (28) this quantity is plotted with respect to c/a in Fig. 4. In this calculation it is convenient to eliminate T in (27) by use of (24) as

$$\frac{\Phi\,\mu}{(1-\nu)ak} = \frac{2\mu(w)_{z=0,\,r=c}}{(1-\nu)ak}$$

$$= \frac{16}{\pi}\,\frac{(1-c^2/a^2)^{\frac{1}{2}}\left\{(1-c^2/a^2)^{\frac{1}{2}} - \frac{c}{2a}\cos^{-1}\frac{c}{a}\right\} - 1 + \frac{c}{a} + \frac{c}{2a}\int_{c/a}^1 \frac{\cos^{-1}\frac{c}{a\tau}\,d\tau}{(\tau^2-c^2/a^2)^{\frac{1}{2}}}}{(1+2\nu)\left\{(1-c^2/a^2)^{\frac{1}{2}} - \frac{c}{2a}\cos^{-1}\frac{c}{a}\right\} + (1-2\nu)} \tag{29}$$

The similar result (12) for a slit and the result for infinite array of slits under anti-plane shear are also shown in Fig. 4. If we consider a critical range of c/a where $\Phi \ge \Phi_{crit.}$ then the penny-shaped crack gives the largest range of the three cases. E. Orowan [10] and G. R. Irwin [11] have proposed that the stress T to propagate a crack of length c in a semi-brittle solid of elastic constant E should be given by a type of Griffith equation

$$T/2 \approx (E\,\gamma/c)^{\frac{1}{2}} \tag{30}$$

in which γ represents plastic work required to increase the area of the crack by a unit amount. A similar expression to (30) may now be obtained from (24) and (28). When $c/a \approx 1$, equation (24) is approximated as

$$T/2k = \frac{\sqrt{2}}{1-2\nu}(1-c/a)^{\frac{1}{2}} \tag{31}$$

and equation (28) is approximated as

$$\Phi \approx \frac{24(1-\nu)k}{\pi(1-2\nu)\mu}c(1-c/a) \tag{32}$$

Eliminating (1-c/a) from (31) and (32), we have

$$T/2 \approx \left\{\frac{\pi\,k\,\mu\,\Phi}{12(1-\nu)(1-2\nu)}\Big/c\right\}^{\frac{1}{2}} \tag{33}$$

4. DISLOCATION SOLUTION FOR PENNY-SHAPED CRACK

It will be shown that the approach using the dislocation distribution and that using the Dugdale plasticity hypothesis are equivalent for the case of the penny-shaped crack.

According to F. Kroupa [12] the stress σ_z due to a circular dislocation located on the plane, z = 0, is

$$\sigma_z = \frac{b\,\mu}{2(1-\nu)}\int_0^\infty s\xi J_0(\xi r)J_1(\xi s)d\xi\,, \qquad z = 0 \tag{34}$$

where b is the Burgers vector in the z direction and s is the radius of the dislocation. Therefore, the stress due to the distribution of dislocations $\alpha_{\theta z}$ inside a circular region $z = 0$ $(0 \le r \le a)$ is

$$\sigma_z = \frac{\mu}{2(1-\nu)} \int_0^a s\, \alpha_{\theta z}(s) ds \int_0^\infty \xi J_0(\xi r) J_1(\xi s) d\xi \,. \tag{35}$$

On the other hand we have from (15) and (19)

$$\sigma_z = -\frac{1}{r}\frac{d}{dr}\int_0^r \frac{t\, f(t)dt}{(r^2-t^2)^{\frac{1}{2}}} \tag{36}$$

inside the same region. By equating (35) and (36), we are led to the following relation between $\alpha_{\theta z}$ and $f(t)$:

$$\alpha_{\theta z} = -\frac{2(1-\nu)}{\mu}\frac{d}{dr}\int_r^a \frac{f(t)dt}{(t^2-r^2)^{\frac{1}{2}}} \,. \tag{37}$$

From equations (15) and (19) we have

$$(w)_{z=0^+} = -(w)_{z=0^-} = \frac{1-\nu}{\mu}\int_r^a \frac{f(t)dt}{(t^2-r^2)^{\frac{1}{2}}} \,. \tag{38}$$

Thus, we have from (25)

$$-\alpha_{\theta z} = \frac{2(1-\nu)}{\mu}\frac{d}{dr}\int_r^a \frac{f(t)dt}{(t^2-r^2)^{\frac{1}{2}}} \tag{39}$$

which gives the same result as (37). This shows the equivalence between the approaches of continuous theory of dislocations and the classical theory of elasticity.

5. DISCUSSION

The maximum shear $\frac{1}{2}(\sigma_z - \sigma_\theta)$ in Section 3 seems unlikely to produce $\alpha_{\theta z}$ type dislocations. On the other hand a preliminary calculation does not admit $\frac{1}{2}(\sigma_z - \sigma_r) = k$ nor $\frac{1}{2}(\sigma_\theta - \sigma_r) = k$ as a useful criterion. The yield condition at the tip of crack will generally be non-linear as in Mises' criterion leading to difficulty in the mathematics. However, it is obvious that σ_z is a primarily important stress component. Barenblatt [3] in fact assumes σ_z as the cohesive force per unit area at the tip of crack by assuming $a \approx c$.

In this section the following boundary conditions are considered instead of (13). The result is to be compared with similar results using the hypotheses of Barenblatt which were obtained by Keer [13] for a

penny-shaped crack.

$$\begin{aligned}
&\sigma_z = T && z = \infty && \\
&\sigma_z = \sigma_{zr} = \sigma_{z\theta} = 0 && z = 0 && (0 < r < c) \\
&\sigma_z = 2k && z = 0 && (c < r < a) \\
&\sigma_{zr} = \sigma_{z\theta} = w = 0 && z = 0 && (a < r)
\end{aligned} \tag{40}$$

The method of Collins is employed again. The function $f(t)$ in (19) becomes, this time,

$$\begin{aligned}
&\frac{\pi}{2} f(t) = T\, t && (0 < t < c) \\
&\frac{\pi}{2} f(t) = T\, t - 2k(t^2-c^2)^{\frac{1}{2}} && (c < t < a) \,.
\end{aligned} \tag{41}$$

The displacement in the z direction is obtained from (15), (19) and (41) as

$$2\mu\,(w)_{z=0} = \frac{4(1-\nu)}{\pi}\left[T(a^2-r^2)^{\frac{1}{2}} - 2k\int_r^a \frac{(t^2-c^2)^{\frac{1}{2}}}{(t^2-r^2)^{\frac{1}{2}}}\, dt\right] \quad (c \le r \le a) \tag{42}$$

$$2\mu\,(w)_{z=0} = \frac{4(1-\nu)}{\pi}\left[T(a^2-r^2)^{\frac{1}{2}} - 2k\int_c^a \frac{(t^2-c^2)^{\frac{1}{2}}}{(t^2-r^2)^{\frac{1}{2}}}\, dt\right] \quad (0 \le r \le c). \tag{43}$$

The conditions

$$f(a) = 0 \tag{44}$$

$$2(w)_{z=0,\, r=c} = \Phi \tag{45}$$

lead to

$$T/2k = (1-c^2/a^2)^{\frac{1}{2}} \tag{46}$$

$$\frac{\Phi\,\mu}{(1-\nu)ak} = \frac{8}{\pi}(1-c/a)\frac{c}{a} \tag{47}$$

Equation (46) is shown by curve I in Fig. 5. The curve is very similar to curve I in Fig. 3. Curve II in Fig. 5 is obtained from the result of Keer where the mean cohesive force per unit area in $c \le r \le a$ is 2k, that is,

$$2k = \frac{1}{a-c}\int_c^a \sigma_z dr \cong T\Big/\left[\frac{\pi}{2}\left(\frac{8}{9}\right)^{\frac{1}{2}}\left(\frac{a}{c}-1\right)^{\frac{1}{2}}\right] \tag{48}$$

and the displacement is parabolic, that is,

$$2\mu(w)_{z=0} = (1-\nu)\pi ka \frac{(1-r/a)^2}{\left(\frac{c}{a}\right)^{\frac{1}{2}}\left(1-\frac{c}{a}\right)} \qquad (c \le r \le a) \qquad (49)$$

The curves I and II in Fig. 5 are very close in the neighborhood of $a/c = 1$. Equation (47) is shown by curve I in Fig. 6. The curve is lower than curve I in Fig. 4, but still higher than curves II and III in Fig. 4 for $c/a \cong 1$. Thus, the same conclusion as Section 3 can be obtained, since usually a crack starts from the neighborhood of $c/a = 1$.

Curve II in Fig. 6 is obtained from (49) by putting $2(w)_{z=0,r=c} = \phi$. The curve is very close to curve I in the neighborhood of $c/a = 1$. This discussion concludes that several theories of fracture employed by Dugdale [4], Barenblatt [3], and Bilby, Cottrell and Swinden [6], are equivalent in brittle fracture.

Another interesting property of ϕ vs. c/a curve can be seen by curve III in Fig. 6. If a pressure p exists inside the crack, equation (47) becomes

$$\frac{\phi\,\mu}{(1-\nu)ak} = (1+p/2k)\,\frac{8}{\pi}\,(1-c/a)\,\frac{c}{a} \qquad (50)$$

Curve III in Fig. 6 is shown for the case of $p/2k = 0.1$. Since the curve is higher than curve I, the unstable domain of c/a increases. This may be applied to the hydrogen embrittlement of ferrous alloys.

The multiplication and propagation of dislocations related to fracture of the crack need further investigation. According to the relation [14] between the time rate of dislocation density tensor and the dislocation velocity tensor

$$\dot{\alpha}_{23} = (V_{2m3} - V_{m23})_{,m} \qquad (51)$$

where 2 and 3 are θ and z directions respectively, and the parenthesis is proportional [15] to the stress deviation,

$$V_{213} - V_{123} \propto S_{33}$$

$$V_{233} - V_{323} \propto S_{31} \qquad (52)$$

In the plastic domain at the tip, $S_{31} = 0$ and $S_{33} = \frac{1}{3}(2\sigma_z - \sigma_r - \sigma_\theta)$. Since $2\sigma_z >> \sigma_r + \sigma_\theta$, the stress component σ_z acts effectively for the multiplication of the dislocations.

APPENDIX

Equation (37) is obtained by equating equations (35) and (36) and integrating them as follows:

$$\int_0^r \frac{t\,f(t)dt}{(r^2-t^2)^{\frac{1}{2}}} = \frac{\mu}{2(1-\nu)}\,r\int_0^a s\,\alpha_{\theta z}(s)ds\int_0^\infty J_1(\xi r)J_1(\xi s)d\xi \qquad (A1)$$

We note the special case of Sonine's first integral

$$\int_0^t \frac{r^2 J_1(\xi r)dr}{(t^2-r^2)^{\frac{1}{2}}} = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} t^{3/2}\,\xi^{-\frac{1}{2}}\,J_{3/2}(\xi t)$$

and consider the left-hand side of (A1) as an Abel integral equation. Performing the indicated inversion we derive

$$t\,f(t) = \frac{\mu}{2(1-\nu)}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^a s\,\alpha_{\theta z}(s)ds\int_0^\infty t^{3/2}\,J_{\frac{1}{2}}(\xi t)J_1(\xi s)\xi^{\frac{1}{2}}d\xi \qquad (A2)$$

But

$$\int_0^\infty J_{\frac{1}{2}}(\xi t)J_1(\xi s)(t\xi)^{\frac{1}{2}}d\xi = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} t\,s^{-1}(s^2-t^2)^{-\frac{1}{2}} \qquad (s > t)$$

$$= 0 \qquad (t > s)$$

Hence f(t) is found as follows:

$$f(t) = \frac{\mu}{2(1-\nu)}\,\frac{2}{\pi}\,t\int_t^a \frac{\alpha_{\theta z}(s)ds}{(s^2-t^2)^{\frac{1}{2}}} \qquad (A3)$$

The right-hand side of (A3) is regarded as an Abel integral equation whose solution is equation (37).

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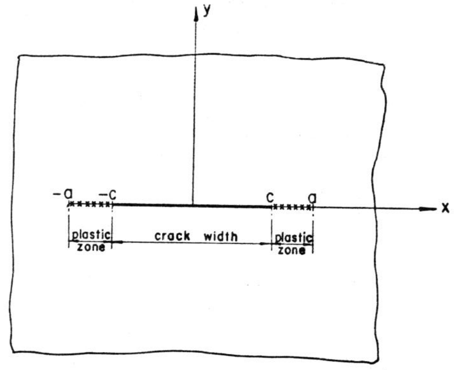

Fig. 1. Slit in anti-plane shear, $\sigma_{yz} = R$ at $y = \infty$.

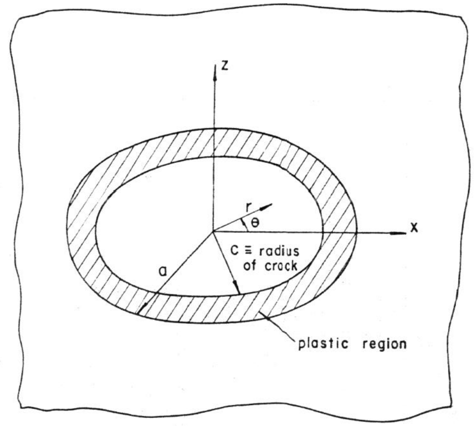

Fig. 2. Penny-shaped crack under uniform tension, $\sigma_z = T$ at $z = \infty$.

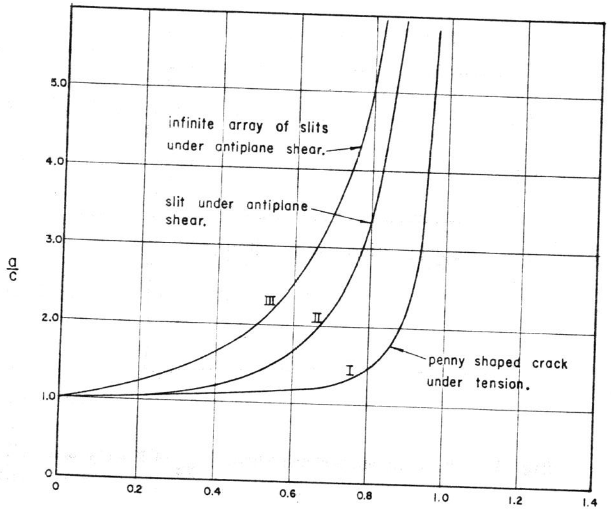

Fig. 3. Length of plastic zone vs. applied stress (T/2k for curve I, R/k for curves II and III).

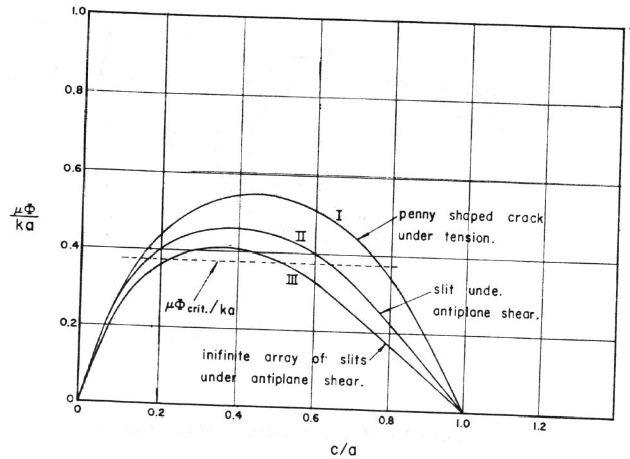

Fig. 4. Plastic displacement at the tip of crack vs. the length of crack.

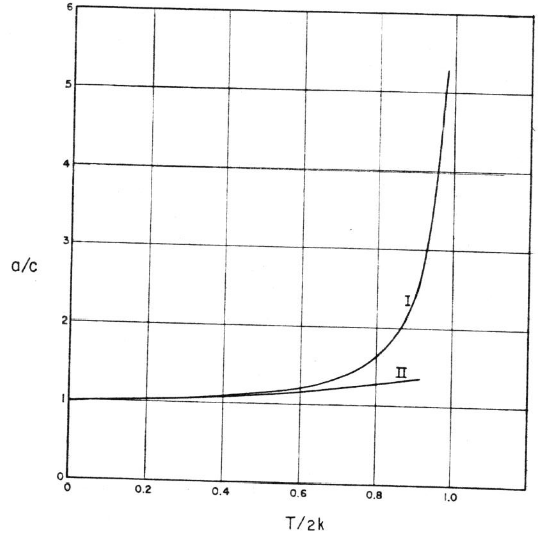

Fig. 5. Length of plastic zone vs. applied stress.

Curve I: $\sigma_z = 2k$ is given in z=0, $c \leq r \leq a$.

Curve II: mean cohensive force per unit area in z=0, $c \leq r \leq a$ is given as 2k.

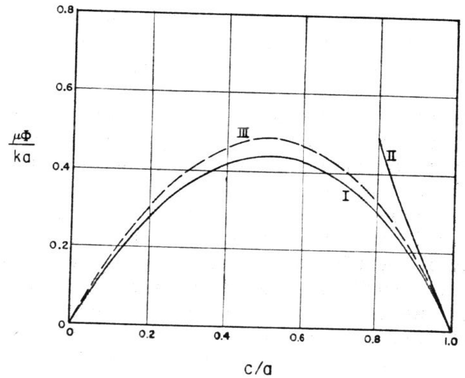

Fig. 6. Plastic displacement at the tip of crack vs. the length of crack.

Curve I: $\sigma_z = 2k$ is given in z=0, $c \leq r \leq a$.

Curve II: mean cohensive force per unit area in z=0, $c \leq r \leq 0$ is given as 2k.

Curve III. (1 + p/2k) times curve I showing the effect of internal pressure inside the crack.