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ON THE GEOMETRICAL APPROACH TO THE MICROMECHANICS OF FRACTURE

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ABSTRACT

The possibility of a micromechanical theoretical approach to the problem of the plastic behaviour of material continua is outlined in terms of Riemannian and non-Riemannian differential geometry. The basic equations for yielding with its extension to cover thermodynamical phenomena and those for the dual yielding are derived, the latter being heuristically treated to afford a possible analytical theory of fatigue fracture.

INTRODUCTION

This is a rather non-analytical exposition of the principal features of our approach to the microscopical mechanics of fracture by differential geometrical methods. Our approach is partly a rearrangement of the same objects treated by the conventional methods with the picture of imperfect crystal lattices etc. By virtue of the different techniques we can also be responsible for different facets of the same problem. The generalization inherent in our standpoint lets the term FRACTURE assume so comprehensive an implication that not only fatigue but also yielding and certain thermodynamical phase changes are unified and brought under the same general heading. To one end of it is connected the dislocation theory of continuous distribution by distant parallelism which originated in Europe and at another as its dual emerges the concept of couple stress which is being given much attention recently.

The material body is treated as an imperfect continuum in which various kinds of anomalies are distributed to be represented by the spaces of connection or generalization thereof studied in Riemannian and/or non-Riemannian geometry. It has recently become well recognized that the dislocation density is mathematically the torsion tensor and the

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incompatibility is the Riemann-Christoffel curvature tensor. However, there is another kind represented by the concept which we call the Euler-Schouten curvature tensor the importance of which should be more emphasized from the standpoint of the mathematical theory of fracture. This is what we have been handling for nearly two decades in connexion with our THEORY OF YIELDING([1], [2], [3], [4], [5]).

The criterion for yielding has been defined analytically as adjoint to the Riemannian metric space equipped with a metric which we call a strain metric of naturalization of the material. A space dual to it, called the stress space, is being given much attention recently. As adjoint to this latter space S. Minagawa assumed a set of equations which correspond dually to the equations of yielding and have the same structure [6]. The dual eigen condition which may be compared to the condition of fatigue fracture, as may the other eigen condition to that of yielding, will be the subject we are concerned with.

The analytical verification and physical meaning of the assumption are afforded by unification of the stress space and the strain space so that the critical condition at which the unified non-Euclidean phenomena set in includes yielding, fatigue fracture and martensitic transformation among others. The rôles of the stress and of the incompatibility are interchanged in the dualism. The temperature has been proved to be equivalent in effect to an isotropic stress. It is suggested that they are also united into parts of one and the same general concept known as the material-energy tensor in general relativity theory.

I. YIELDING AND ITS GENERALIZATION

1. The Geometrical Terminology and the Multidimensional Picture of Yielding

The suggestion to the multidimensional picture of yielding was originally by an analogy with the buckling of flat plates, where plastic disturbances were compared to a certain non-Euclidean behaviour of the material manifold so that they are represented by deviations from the three-dimensional flat plate [1], [2]. In the subsequent investigations continued to the present time, the validity of this assumption was brought under more analytical and physical light. It is, therefore, not merely an intuitional postulate, as was sometimes skeptically inferred to, but upon the basis of more logical, even pragmatical scrutiny, that we reconfirm the fundamental criterion of yielding summarized into a set of differential equations, which are in the representative case ([2], [3], [4], [7] etc.),

$$\partial_\nu \partial_\lambda (B^{\varkappa\lambda\mu\nu} \partial_\mu \partial_\varkappa w) - \partial_\lambda (\sigma^{\varkappa\lambda} \partial_\varkappa w) = 0 \tag{1}$$

$$B^{\varkappa\lambda\mu(\nu)} \partial_\mu \partial_\lambda w = 0$$
$$\partial_\mu (B^{(\nu)\varkappa\lambda\mu} \partial_\lambda \partial_\varkappa w) - \sigma^{\varkappa(\nu)} \partial_\varkappa w = 0 \tag{2}$$

where $x^\varkappa$ ($\varkappa$=1,2,3) are rectangular coordinates of the material point in undisturbed state, and (ν) means the direction of the surface normal. $\sigma^{\varkappa\lambda}$ is the applied stress tensor, w a scalar or scalars depending on $x^\varkappa$, and $B^{\varkappa\lambda\mu\nu}$ the tensor of the material property by which the resistance to plasticity is represented at yielding. The set consists of the field equation (1) satisfied in the interior of the specimen and of the boundary condition (2) for the free surface. Except the dimension number, they are entirely the same as the equations of buckling of a plate which is free at the edges.

The homogeneous character of these equations should be noted since they need to define a characteristic condition at which only the small disturbance w can set in. Thus the stress standing in them needs to be restricted so that the specimen goes into a sudden change when its characteristic value is arrived at. Such is the case either for buckling or yielding.

The quantity w can be multidimensional so that it has a number of components or consists of more than one scalar w^Λ (Λ = 1,2,3...) Its geometrical meaning can easily be looked up in established concepts in differential geometry. In the present case of small disturbances, it is connected with the Euler-Schouten curvature tensor ([8],[4],[9])

$$H_{\mu\lambda}^{\cdot\cdot\Lambda} = \partial_\mu \partial_\lambda w^\Lambda \tag{3}$$

where Λ can be restricted to ≠1,2,3 by taking account that the direction of the meaningful (i.e. incompatible) deviation w^Λ is normal to the original manifold ([4],[9]). In this picture the deformed three-dimensional material is thought to be immersed in a Euclidean space of more than three dimensions and its relative curvature to the latter is described by $H_{\mu\lambda}^{\cdot\cdot\Lambda}$. Therefore, the deformation at yielding may analytically be compared to the swelling out of a three-dimensional manifold, initially flat, into a curved state in a multidimensional space, which happens to be a generalization of the swelling out at buckling of a two-dimensional flat plate into a curved one in three-dimensional space.

With this analytical feature one can also associate what the curved continuous manifold physically signifies, taking account that it is a picture not in the ordinary three-dimensional space but in a higher one. Being the labels of the initial location of the material point, the coordinates used in describing it can no longer agree with the actual location after the yielding accompanied of necessity by an incompatible deformation.

It is well known that from the Euler-Schouten curvature tensor is constructed the Riemann-Christoffel curvature tensor of a Riemannian space

$$R_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa} = \sum_{\Lambda} H_{[\nu}^{\cdot\kappa\Lambda} H_{\mu]\lambda}^{\cdot\cdot\kappa} \tag{4}$$

so that the latter is always accompanied by the former but the converse does not always hold (see [3],[8],[9], etc.). Therefore, the relative curvature $H_{\mu\lambda}^{\cdot\cdot\Lambda}$, or the distribution of w^{Λ} is a measure of a kind of anomaly or generalized incompatibility

These are also summarized into a more general concept of holonomy group propounded by French geometre Elie Cartan [10]. Its physical meaning is that, wherever these curvature tensors appear, the connexion between the elements of the material manifold is no longer the same as it was initially before the anomalous deformation.

The concept of holonomy group comprises also a discrepancy due to intervention of another kind of anomaly which is analytically represented by Cartan's torsion tensor [11]. That it can be reinterpreted as the crystallographical dislocation density in continuum terminology has been pointed out by us [3], as well as by some others ([12],[13]) so that it has become an object of interest of the metallurgist ([14], [15] etc.).

In Cartan's terminology these anomalies correspond to the discrepancies connected with a small circuit in the plastic manifold, such as are obtained by connecting each consecutive pair of elements on it. In others words, the anomaly is measured by the discrepancy at the cut when the infinitesimal local topology is preserved everywhere else while the metric is deformed. There can so come about a discrepancy of location corresponding to the torsion and those of orientation corresponding to the curvatures.

The relations between the deformed and undeformed manifolds are anholonomic so that the anholonomic object, known in differential geometry ([3],[16],[17]), occupies an important rôle in the recognition

of plastic deformation [18]. It is to associate particular sets of anholonomic coordinates to particular deformation conditions. One goes from mathematics to physics, from a macroscopic to a microscopic, which is almost the same as atomistic, observation, at this.

To use an anholonomic picture, as well as that of holonomy group, is to tear plastically the connected manifold into Pfaffian elements [19], [20]. The curvature-and-torsion is a résumé of the torn state.

At this we need to emphasize that the language of differential geometry and that of crystal lattice are approximation of each other. Which of them is more accurate needs to depend on the nature of the problem and the scale of the phenomena we are concerned with (cf. [21]).

If we use a more global scale some microscopic anomalies are statistically lumped into a large scale one. Conversely, a more global anomaly may split into more than one unit of a more microscopic kind [22]. Such is the case in the multidimensional picture of yielding where the torsional anomaly, which is more microscopic, apparently disappears being lumped into a curvature tensor by an inverse process of tearing. Should one argue that the multi-dimensional picture of yielding neglects the dislocations in the specimen, one would miss the most subtle point in this connexion.

One more remark should be made in connexion with the foregoing. Here is inherently the origin of the concept of the Cosserat Continuum, the study of which is recently fashionable. It is defined as a continuum susceptible not only of the ordinary stress but also of the so-called couple stress. As has been well known as Boltzmann's axiom, the couple stress originates when the volume element of the continuum is not reduced to zero, which means that one cannot penetrate the structure of the mechanics in the element. It is non-local in a broad sense (cf. CONCLUSION). Its volume can be large enough to allow a couple, of stress (if force or couple is concerned about), or a pair of dislocations if the deformation is concerned about. The resistance to the grouth of the Euler-Schouten curvature tensor has in fact the character of a couple as the analogue of a couple on the cross section of a plate.

2. Origin and Structure of the Material Constants $B^{\varkappa\lambda\mu\nu}$ and Material-Energy Tensor

The physical meaning of the material tensor $B^{\varkappa\lambda\mu\nu}$ has been explored generally as well as in close connexion with the analysis of several different types of yielding. Whatever the physics so concerned, it originates statistically as the average property of the more microscopically non-uniform structure of the material. The question is to find the elements or objects which one can subject to the processes of statistics. As we work with the differential manifold, such elements need also be given in terms of differential geometry.

In statistical mechanics they are atoms or molecules or elementary particles under classical or quantum-mechanical laws. These particles describing geodesics of their respective definitions, the objects of plasticity statistics could be geodesics in the non-Euclidean manifold. One can therefore start with a statistics of continuous distribution of geodesics, or the set of geodesic deviations from a given geodesic. This is brought to the thermodynamical criteria in variational formulation. By some assumption, such as local statistical isotropy or its generalization one can reduce it to a considerably simple form generalizing in a sense that of the principle of general relativity (cf. [24]). The material constant B originates as coefficients of the terms in the Ricci tensor or Einstein tensor. Whilst the stress terms correspond to those assigned to the material-energy tensor as its three-dimensional degeneration.

Strict isotropy needs to be excluded for the deformed manifold since it would annull all components of $B^{\varkappa\lambda\mu\nu}$. But the material and resulting non-vanishing tensor $B^{\varkappa\lambda\mu\nu}$ can still be statistically isotropic. The analysis so far carried out on an isotropic material can be meaningful [25].

From the foregoing is also obtained then an important suggestion as follows. The stress can be replaced by a more general material energy-tensor such as related to the thermal effects. This is thermo-dynamically anticipated since the pressure stands on the same footing as the absolute temperature multiplied by an entropy factor. Thus the formula for yielding is extended to cover some thermodynamical critical phenomena like martensitic transformations.

3. Analysis of Yield Points and Martensitic Transformation

The manner of yielding naturally varies with the boundary condition and the stress distribution. We have some representative

cases studied elsewhere, considerably in detail. We shall not repeat most of them, e.g.,

i) Tresca's classical theory of maximum shear for an isotropic material [26], [27], and

ii) also von Mises' criterion reached by introducing some statistics from Tresca's [28].

iii) Apparently incomprehensible features of the empirical facts that the yield type depends much on the stress distribution are explained ([27], [29]).

We shall here present only some minor amendments in regard to quantitative results of

(a) The problem of some of the apparently phenomenological constants standing in the equations of yielding, shown to be derived somewhat indirectly from the elastic constants [29], and

(b) Martensitic transformation and hysteresis there of [30].

(a) <u>This is concerned with Poisson's ratio</u>. By statistical approach, one can argue that the constants $B^{\varkappa\lambda\mu\nu}$ have <u>the six-index</u> structure:

$$D^{\hat{i}\bar{j}\bar{k}\hat{\ell}\bar{p}\bar{q}} = -\frac{1}{2} C^{\hat{i}\bar{j}\bar{k}\hat{\ell}} dx^p dx^q$$

where $C^{\hat{i}\bar{j}\bar{k}\hat{\ell}}$ are elastic constants, the bar indicating the average and ˇthe fluctuation [7]. The indices $\bar{j}$, $\bar{k}$, $\bar{p}$, $\bar{q}$ are reduced to the three dimensional space and $\hat{i}$ and $\hat{\ell}$ to those normal to it. Taking account of the symmetry requirement of $C^{\hat{i}\bar{j}\bar{k}\hat{\ell}}$ and of the construction, the formula can further be simplified.

If the material is isotropic, the elasticity tensor is reduced to Young's modulus E and Poisson's quantity ν. As the result we have

$$D^{\hat{i}\bar{j}\bar{k}\hat{\ell}\bar{p}\bar{q}} = \varpi^2 E' \begin{bmatrix} 1 & \nu' & \nu' \\ \nu' & 1 & \nu' \\ \nu' & \nu' & 1 \end{bmatrix} \qquad (6)$$

where

$$E' = \frac{1-\nu}{1+\nu} \cdot \frac{E}{1-2\nu} , \qquad \nu' = \frac{\nu}{1-\nu}$$

and ϖ^2 has the dimensions of area.

If it is a priori known that the disturbance is two-dimensionally restricted, we have, in place of (6),

$$D^{\hat{i}\hat{j}\bar{k}\bar{l}\bar{p}\bar{q}} = \varpi^2 \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix} \tag{7}$$

On the other hand, if $\hat{i}$ and $\hat{l}$, the directions of which are directed to the outside of the three-dimensional space, are surpressed or fixed, the isotropy requires

$$D^{\hat{i}\bar{j}\bar{k}\hat{l}\bar{p}\bar{q}} = B\begin{bmatrix} 1 & -\kappa & -\kappa \\ -\kappa & 1 & -\kappa \\ -\kappa & -\kappa & 1 \end{bmatrix} \tag{6'}$$

which should be compared with (6). The Poisson's quantity is thus carried over to plasticity where it is multiplied by $-1/(1-\nu)$ or -1. For an ordinary material such as mild steel, we have $\nu = 1/3$, so that

$$\kappa = -\frac{1}{2} \quad \text{or} \quad -\frac{1}{3}$$

according as the disturbance is three or two-dimensional.

(b) The analysis of martensitic triggering has been carried out assuming isotropy for the material and replacing the stress by the temperature factor [36]. The field equation is then simplified to

$$B\Delta\Delta w + \eta T \Delta w = 0 \tag{8}$$

where B is a constant factor, T is the absolute temperature, and η is a factor which is an isotropic summary of the entropy criterion. The specimen is assumed to be sufficiently large so that

$$w = \text{constant}$$

at infinity.

Equation (8) is naturally reduced to one of the Helmholtz type

$$\Delta\psi + g\psi = 0 \tag{8'}$$

where

$$\psi = \Delta w,$$
$$g = \frac{\eta}{B}T$$

The martensite field is assumed to have either a plate- or needle-form embryo so that the oblate or prolate spheroidal coordinates can be used. For merely analytical purposes, it is convenient to modify the distribution of w at infinity so that the rigorous Helmholtz equation is replaced by an approximation which admits the procedures of separation of variables in spheroidal coordinates.

The analysis is still too much complicated to be recapitulated in this restricted space. According as the embryo is a needle or disc, normal or exaggerated, various configurations are obtained. It is remarkable that a certain lower bound is assumed for the meaningful range of linear dimension. It could be the lattice constant. This also has been the case in our theoretical derivation of Tresca's criterion.

The result of the analysis is plotted in term of the parameter

$$J = gh^2$$

where h is the lower bound of the meaningful length, in Fig. 1 which may apparently explain the theoretical possibility of the martensitic hysteresis.

II. A HEURISTIC APPROACH TO THE PROBLEM OF FATIGUE

4. Dual Standpoint and Dual Equations applicable to Fatigue Fracture Problems

This is based on the recognition of the stress metric by which the famous Beltrami stress functions are defined (see [31] and [32]). This is dual to the strain metric. The corresponding dual dislocation and dual compatibility are given much attention recently, as by S. Amari [31].

The stress tensor emerges as the Einstein tensor of the stress space as does the incompatibility tensor as one of the strain space. If they are denoted respectively by

$$I^{\kappa\lambda} \quad \text{and} \quad I'^{\kappa\lambda}$$

and the strain and stress tensors by

$$e_{\lambda\kappa} \quad \text{and} \quad e'_{\lambda\kappa} .$$

The metric tensors are

$$a_{\lambda\kappa} = \delta_{\lambda\kappa} + 2e_{\lambda\kappa} ,$$

and

$$a'_{\lambda\kappa} = \delta_{\lambda\kappa} + 2e'_{\lambda\kappa} .$$

Then either

$$V = (e_{\lambda\kappa} \mid I'^{\kappa\lambda}) \stackrel{\text{def}}{=} \frac{1}{2}\int I'^{\kappa\lambda} e_{\lambda\kappa}\, dX ,$$

or

$$V' = (e'_{\lambda\kappa} \mid I^{\kappa\lambda}) \stackrel{\text{def}}{=} \frac{1}{2}\int I^{\kappa\lambda} e'_{\lambda\kappa}\, dX$$

has the dimensions of energy, where $dX = dx^1 dx^2 dx^3$ in rectangular coordinates. These are part of the formulae for the mutual interaction between the two spaces or two facets of the plasticity, one related to the stress and one to the strain.

One can analytically amalgamate the two Riemannian spaces into one without being concerned about its realization in ordinary space because it is a non-Euclidean concept. Thus we have the resultant metric tensor

$$(\phi| = (e_{\lambda\kappa}| + (e'_{\lambda\kappa}|$$

and the resultant Einstein tensor

$$|\Phi) = |I^{\kappa\lambda} + I'^{\kappa\lambda} + Y^{\kappa\lambda})$$

where $Y^{\kappa\lambda}$ is the interaction term. The resultant energy, owing to the self- and mutual coupling, is given by

$$W \stackrel{\text{def}}{=} (\Phi \mid \phi) = \frac{1}{2}\int \phi\Phi\, dX$$

The energy is subject apparently to the first and second thermodynamical laws, which is either

$$\delta U \stackrel{\text{def}}{=} \delta W - \delta V > 0 \quad \text{or} \quad \delta U' \stackrel{\text{def}}{=} \delta W' - \delta V' > 0$$

where

$$\delta U \stackrel{\text{def}}{=} \delta(V' + \cdots) \quad \text{or} \quad \delta U' \stackrel{\text{def}}{=} \delta(V + \cdots)$$

Evidently, δU or $\delta U'$ takes over the rôle of the lost energy and δV or $\delta V'$ is the increment of the free energy.

No energy is supplied in the case of static equilibrium so that we have the variational criterion $\delta W = 0$ or

$$\delta(U + V) = 0 \tag{9}$$

or

$$\delta(U' + V') = 0 \tag{9'}$$

Taking account of the small disturbance at the critical point, we may preserve only the principal significant terms.

If the stress-functions and hence the stresses remain unvaried, the dual geometrical invariant $e'_{\lambda\varkappa}$ and $I'^{\varkappa\lambda}$ are precluded from the variation. This assumption corresponds to (9) and we have

$$\frac{1}{2}\int k^{\varkappa\lambda}\delta K_{\lambda\varkappa}dX + \frac{1}{2}\int M^{\varkappa\lambda}\delta e_{\lambda\varkappa}dX = 0 \qquad (10)$$

where $k^{\varkappa\lambda}$ and $M^{\varkappa\lambda}$ are appropriate functions of $e^{\varkappa\lambda}$, $e'^{\varkappa\lambda}$, $I^{\varkappa\lambda}$, $I'^{\varkappa\lambda}$ etc. Let this be called the first assumption.

If the strain and hence also the incompatibility remain unvaried, the geometric invariant $e_{\lambda\varkappa}$ and $I^{\varkappa\lambda}$ are precluded from the variation. This assumption corresponds to (9') and we have

$$\frac{1}{2}\int k'^{\varkappa\lambda}\delta K_{\lambda\varkappa}dX + \frac{1}{2}\int M'^{\varkappa\lambda}\delta e'_{\lambda\varkappa}dX = 0 \qquad (11)$$

where $k'^{\varkappa\lambda}$ and $M'^{\varkappa\lambda}$ are appropriate functions of $e'^{\varkappa\lambda}$, $e^{\varkappa\lambda}$, $I'^{\varkappa\lambda}$, $I^{\varkappa\lambda}$ etc. Let this be called the second assumption.

It is obvious that, in this rough approximation, the coefficients $k^{\varkappa\lambda}$, $M^{\varkappa\lambda}$, $k'^{\varkappa\lambda}$, $M'^{\varkappa\lambda}$, may be replaced by their mean values when only the mean characteristics over a considerable space in the material is concerned about. As far as we handle the problem of small deviation of strain and incompatibility from the Euclidean condition, the terms in $e^{\varkappa\lambda}$ etc. can be neglected so that we assume

$$k^{\varkappa\lambda} \doteq e'^{\varkappa\lambda} - \frac{1}{2}\delta^{\varkappa\lambda}e' \quad , \qquad e' \overset{\text{def}}{=} \delta_{\nu\mu}e'^{\mu\nu}$$

and

$$M^{\varkappa\lambda} \doteq I'^{\varkappa\lambda} \quad = \text{stress tensor}.$$

Dually for the problem of small deviation of stresses and stress functions, we assume

$$k'^{\varkappa\lambda} \doteq e^{\varkappa\lambda} - \frac{1}{2}\delta^{\varkappa\lambda}e \quad , \qquad e \overset{\text{def}}{=} \delta_{\nu\mu}e^{\mu\nu}$$

and

$$M'^{\varkappa\lambda} = I^{\varkappa\lambda} = \text{incompatibility}.$$

The variational criterion under the first assumption or approach by the strain metric then leads to the equations of yielding treated in the foregoing. Dually under the second assumption or approach by the stress metric it needs to leads to the dual equations intuitively assumed to hold by S. Minagawa [6], which are

$$\partial_\ell\partial_i(B^{ijk\ell}\partial_k\partial_j w') - \partial_i(I^{ji}\partial_j w') = 0 \qquad (12)$$

$$B'^{ijk(\nu)}\partial_k\partial_j w' = 0$$
$$\partial_\ell(B'^{(\nu)jk\ell}\partial_k\partial_j w') - I^{i(\nu)}\partial_i w' = 0 \qquad (13)$$

where w' is dual to the w in the foregoing, and i, j, k, ℓ = 1,2,3.

Let us call them the equations of dual yielding ([34],[35]).

5. Baptizing Fatigue Fracture as Dual Yielding

The dual yielding needs to occur when the incompatibility amount reaches a certain limit by any preliminary mechanical processes. The limiting value is assigned by the structure of the equations (12) and (13), the latter of which may change to some extent with the condition of the boundary and the shape of the specimen. But such critical mechanical phenomena as compatible with dual yielding are very few. Perhaps, the fatigue fracture could be the sole one as was anticipated by Minagawa. We need therefore to investigate whether this is essentially so equipped or not.

The fatigue fracture can be regarded as achieved through two consecutive processes; one is the growth of a certain genesis of fracture, and the other is its development into visible size.

The repeated stressing at the fatigue experiment seems to make the growth of the genesis easier at each step. An atmosphere which carries this tendency to genesis may be assumed gradually to be produced in the interior of the material of the specimen and to grow until a limit is reached. If the limit is eventually the dual yielding, the increasing atmosphere cannot mean but the growing incompatibility. What is then the genesis?

At each cycle of load, something unobserved from the outside is added so that the interior condition is altered without altering the external appearance. There are two kinds of physical geometrical objects that cannot directly be mechanically observed. One is the stress-function metric that does not yield an observable stress or Riemann-Christoffel curvature tensor. Another is the incompatibility. Both are not directly observed from the outside but play dominant rôles in defining the criterion of dual yielding.

Let us try to substitute these significant Riemannian geometrical terms in the tentative proposition:

When a growing atmosphere reaches a certain amount a genesis begins to develop into visible amount [6].

Evidently the atmosphere needs to have a critical value at which a stability limit should be reached hence it must be set in correspondence with the incompatibility which has a limit in the theory of dual yielding. There remains then the stress-function as an unobserved object to be set in correspondence with the genesis. Its non-Euclidean part is obviously included in the dual strain $e'_{\lambda\kappa}$. Therefore, the unobserved genesis is represented by a scalar disturbance w' (or more general functions) corresponding dually to the disturbance function w (or more general functions) in the theory of yielding.

That the unobserved change of the metric due to the intervention of w' does not necessarily mean an observable change of the stress is dually in correspondence with the possibility of an unobserved w, either yielding a Euler-Schouten curvature tensor without an observable Riemann-Christoffel curvature tensor, i.e. a stress or an incompatibility.

If the above correspondence is justified the fatigue fracture is no doubt a dual yielding. However, a straightforward objection could arise such as follows: The equation of yielding which does not involve any instationary terms depending on time, does not seem to be answerable for fatigue which involves of necessity instationary changes owing to the cyclic loading. The objection could be excluded if one does not fail to recognize the analytical structure of the origin of the criteria of yielding and of dual yielding as a kind of stability limit. The basic variational criterion came initially of a set of non-homogeneous equations from which the homogeneous part

has been extracted to account for the critical condition of yielding. The non-homogeneous part so subtracted consisted of the energy-momentum tensor summarizing the pre-existing distributions of dislocations dual dislocations etc.

Similarly, there can be both stationary and instationary dual energy-momentum tensor summarizing the disturbances of dual dislocation etc. including repeated stresses and instationarily changing moment-torque stresses.

However, it is question whether the mean incompatibility adopted in the homogeneous dual equations is so remarkably dependent on time as prohibits the use of the equations of stationary form.

The incompatibility being the atmosphere that grows with the repeated stressing, a positive increment should accumulate at each cycle. The periodically changing quantity, of which the accumulation occurs, needs to take positive values more often and more dominantly than it takes negative values [cf. Fig. 2 (a)], so that the resultant accumulation by interaction increases almost monotonically [Fig. 2 (b)].

Experimentally the fatigue characteristics are described in reference to the number of cycles alone, the atmospheric incompatibility not remarkably depending on the minor difference of time or phase in a cyclic step. So we can conclude that too detailed an instantaneous deviational feature of the atmospheric condition of the fatiguing specimen need not be considered in using the dual equations of yielding.

The proposed equation of dual yielding can therefore be used to analyze the fatigue limit without essential objections. The limit can be calculated without being too much concerned about the external instationary appearance.

6. Fatigue Analysis for Isotropic Disturbance

We shall show that the S-N curve (S standing for the stress amplitude and N for the number of cyclic loadings) for fatigue limits is obtained generally under some assumption from the foregoing.

Without much objection, the material may be assumed to be plastically isotropic. Moreover, there being no orientation for the incompatibility tensor, one may also assume it to be isotropic. This assumption is due to Minagawa [6]. In such a case the dual field equation can be simplified to

$$C(\Delta\Delta w' - N\Delta w') = 0 \qquad (14)$$

which has the same form as (8), where N is proportional to the incompatibility and C a constant [27].

Particular solutions of (14) can be obtained in the same manner as studied elsewhere in regard to the yield criterion under hydrostatic pressure. Usually some disturbance might appear on the surface of the specimen, but it may not penetrate deep. The deeper interior seems to be subject to another kind. Such is afforded by an analogy of the locally isotropic disturbance for yielding.

The assumption of local isotropy may also be brought over to the dual yielding provided that the dominating disturbance therefor is not affected by the presence of the boundary, which seems to be the case.

Entirely in the dual manner to the analysis of yielding under the assumption of local isotropy, we obtain

$$N\sqrt{S} = 0 \qquad (15)$$

where S is the Riemannian curvature of the stress space whence either

$$\text{i) } N = 0 \qquad \text{or} \qquad \text{ii) } S = 0. \qquad (16)$$

Contrary to the case of yielding, these criteria

i) and ii) are not altogether trivial in the case of dual yielding or fatigue fracture. Taken together, they appear as the set of thick lines, i) on the vertical axis and ii) on the horizontal axis of coordinates, where the material needs to fracture (Fig. 3). The abscissa N is proportional to the increasing incompatibility or atmosphere or the number of cycles. The ordinate is the Riemannian curvature which has the dimensions of stress. But the latter's relation to the fluctuating stress impressed by the cyclic load is not clear. The theory summarized in i) indicates only that a certain isotropic stress field could be increased without limit if the incompatibility is absent. The second condition ii) indicates that the limit is imposed and reduced to zero load (amplitude of cyclic stress) as soon as the load is repeated any finite number of cycles.

The practical situation could be a more gradual change from i) to ii) such as indicated by the dotted line in Fig. 3 [cf. formula (16)]. Such may happen if (15) does not rigorously hold so that the right-hand side takes a finite value in place of zero. This is the primary picture of fatigue assumed by Minagawa.

The experimental relation between the S and N is illustrated in Fig. 4, where a horizontal asymptote is shifted to a distance from the horizontal axis. The amount is called the endurance limit.

We need to explain the finite endurance limit. We have been prepared for this by our summarizing various physical quantities into the same class of material-energy tensor standing in the same place as the stress tensor in the criterion for yielding or of dual yielding. Therefore, the N and S in the formula (15) can be replaced by

$$N - N' \qquad \text{and} \qquad S - S'$$

respectively where S comes from the ordinary stress and S' from the other stress equivalents; similarly for N and N'. The formula (15) is thus modified to

$$(N - N')\sqrt{S - S'} = 0. \qquad (16)$$

The dual pair N and S may be combined into one in the range where the amalgamation of the dual spaces is most remarkable. Part of -N' then has the character of varying S and part of -S' that of varying N. Hence (16) can be refined into

$$(N + \lambda S - N_0)\sqrt{S + \mu N - S_0} = 0 \qquad (16')$$

for such a range where the coefficients λ and μ are appropriately assumed and S_0 and N_0 can be constants in ordinary cases. For extremal ranges where one of S and N does not vary, or grows infinitely, it goes into

$$N = N_0 \qquad (\lambda S = 0) \qquad (16'')$$

or

$$\sqrt{S - S_0} = 0, \quad (\mu = 0). \qquad (16''')$$

As N varies, in general features, these branches connected into one such as is represented in Fig. 5 where

$$\frac{N_0}{\lambda} = S_0 > 0$$

is assumed. The significant branches are along PA, PB and BS. Fracture cannot take place near N=0 except for a large extremely S so that the branch PC is excluded. Similarly the branch CD. The branches EF and P'B' are made insignificant by the precedence of PB in the process of increasing N.

Connecting these significant regions, we obtain finally the lower limit of the region of rapture points which resembles the empirical ones.

CONCLUSION

We have gone briefly over the basic features of the analysis of the extended fracture problems covering yielding, fatigue, martensite triggering and its hysteresis. Those other empirical features which are not so explained may not yet be known to us in sufficiently a consolidated form even experimentally.

It may be worthwhile to draw attention on this occasion to the prevalent misunderstanding of the rôle of the methods of differential geometry applied to these problems.

The geometry is one of the admissible languages. It is perhaps the most convenient and powerful one as the means of natural philosophy. But one language is seldom perfectly translated into another. Even if, on applying tensorial geometry, our investigation might fall into a trap which dislocation theory can circumvent, a new direction of many an unexpected penetration can thereby be opened. As one of the remarkable characteristics may be mentioned the fact that the result of statistical observation finds often a most natural and appropriate expression in the language of non-Euclidean differential geometry. The statistical handling of polycrystalline material structures, in connexion with the theory of micromechanics of imperfect continua, needs to provide the valuable clue as a simplified model for the study of higher order anomalies.

An extension of the foregoing will touch upon the problem of what rôle will be played by the extended metric space. Here is expected the possible future penetration to the higher microscopical anomalies which dominates in the real world. Although the details of them may not be noticed clearly as yet, one may cite some evident cases known so far as follows.

On the natural direction in extension of the Riemannian space concept, we find those higher order spaces associated with the celebrated names of Finsler [34] and Kawaguchi [35], [36]. They originate when more non-local features of the anomaly units are considered ([39],[40],[41]), to cover not only the simply local, but also the bi-, tri-, quadri-local cases etc. Such is along the line of the recent non-local theory of elementary particles and can account also for the non-local plasticity theory (e.g. [42]; see also [43]). Part of the endeavour by an apparently different school lies in a certain sense along the dual line to this and is concerned with essentially the extension of the concept of Cartan's area space [44].

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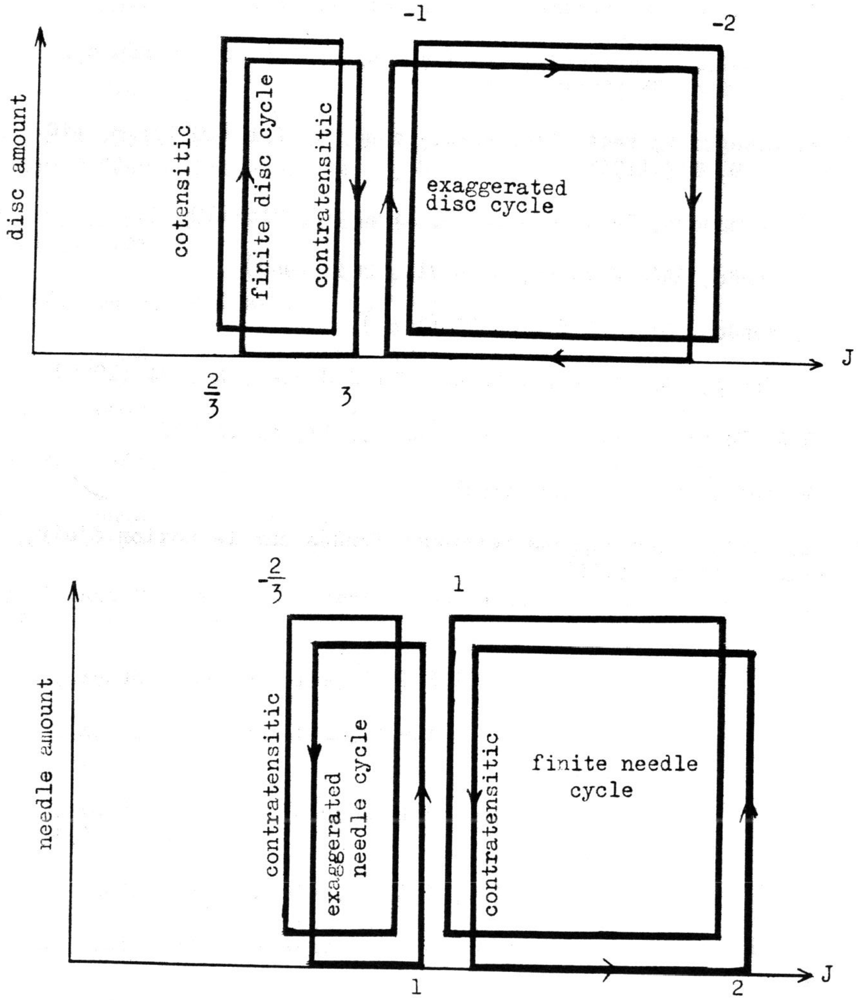

Fig. 1

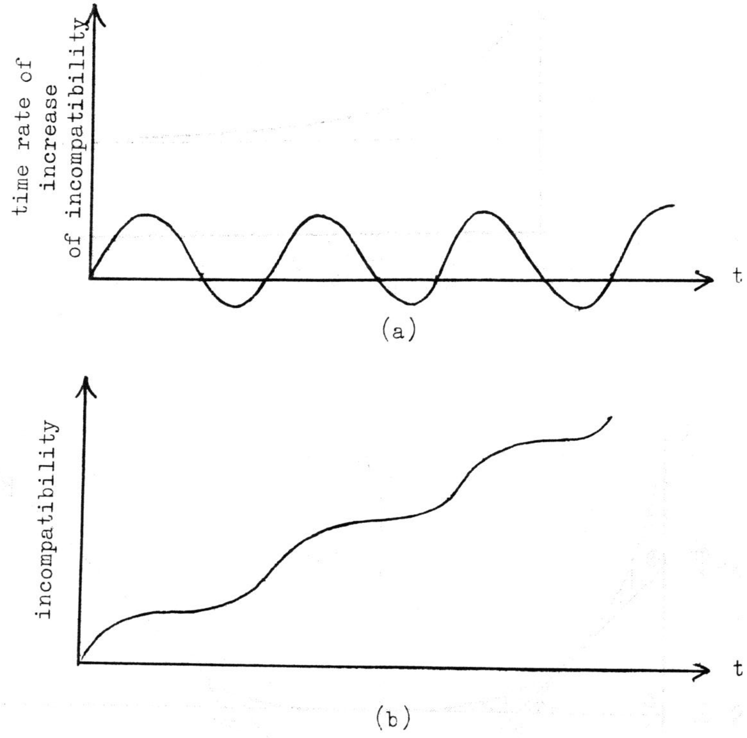

Fig. 2

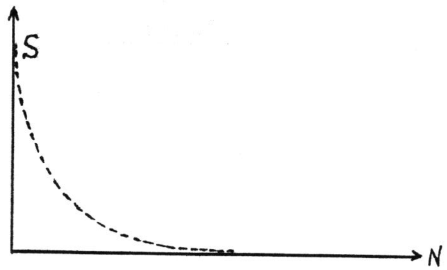

Fig. 3

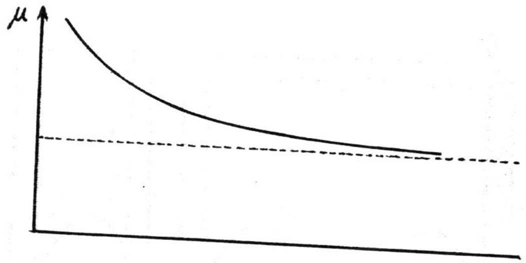

Fig. 4

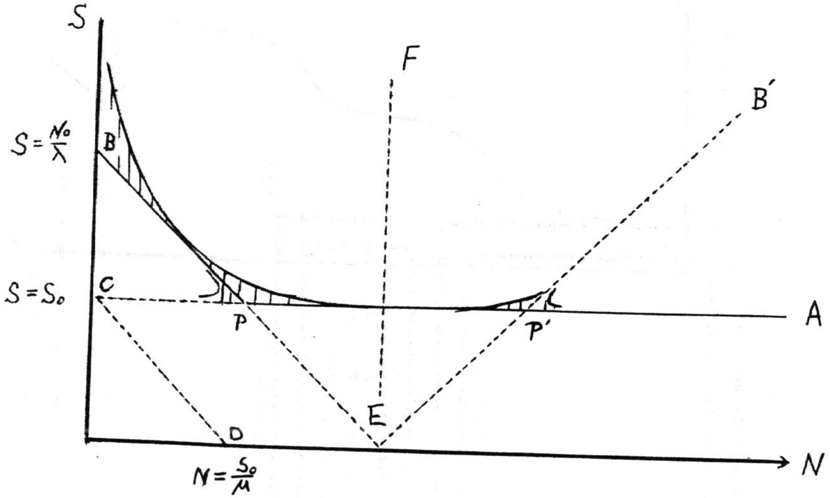

Fig. 5