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On the Concepts of Dual Incompatibility and Dual Dislocation in Relation to Fracture Problem

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Abstract The physical state of material including plastic defects has been clarified by non-Riemannian plasticity theory. It is known that the strain defines the metric character of the plastic manifold and the dislocation distribution defines the torsion character. Here we construct the dual plastic manifold in such a manner that the structural quantities of the dual space are defined by the physical reactions resisting growth of the corresponding structural quantities of the primal plastic manifold. This is proved to be the non-Riemannian generalization of Schaefer-Minagawa's stress space, and its relation to the strain space is clarified by using the concept of dual dislocation.

1. Introduction

This is part of the unifying study in establishing Riemannian and non-Riemannian theory of plastic continua. The plastic state of material can be represented by either of the following two kinds of spaces. One is the strain space of which metric tensor is defined with reference to the strain and the other is the stress space of which metric is defined with reference to the stress-function tensor in Beltrami's sense. The geometrical theory of continua started with imbedding the Riemannian strain space in multi-dimensional space [1], deriving the equation of yielding. Then the non-Riemannian structure was introduced to the strain space to clarify the dislocational structure. The strain space has now been extended to Finslerian space [2] in order to take ferromagnetic structures into account, to four-dimensional one [3] in order to take moving dislocations into account, and to more higher-order spaces [4].

Recently the stress space has taken much attention. The equation of fatigue fracture has been derived by imbedding the Riemannian stress space in multi-dimensional space [5] in entirely the dual manner as that of the strain space. The non-Riemannian structure has also been introduced to the stress space taking account of the torque stress [6], [7], [8]. The existence of the torque stress has also been discussed in relation to the dislocation field [9].

Here, we try to construct the non-Riemannian stress space in entirely the dual manner as the construction of the non-Riemannian strain space. We construct the stress space in such a manner that the structural quantities of the stress space represent the physical reactions caused by the existence of the corresponding quantities of the strain space. This space can be proved to be a non-Riemannian generalization of Schaefer-

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Minagawa's stress space [10], [11]. In this connection, the concepts of dual dislocation and dual incompatibility are introduced and their relations to primary quantities are investigated. The geometry of non-Riemannian stress space will clarify the mutual relations of the dual quantities.

Let us start with the ordinary definition of the strain space.

2. Non-Riemannian strain space [12]

The material manifold has been deformed by virtue of the plastic defects existing in it. When all the plastic defects are removed by tearing the material manifold, a small material vector dx^i is deformed to $(dx')^i$. There can be assumed the linear relation

$$(dx')^i - dx^i = \beta_{ji} dx^i . \tag{1}$$

Here it is assumed that the deformations are small, and hence the difference between the covariant and the contravariant characters of indices are disregarded. Einstein's summation convention is assumed throughout.

The plastic and elastic quantities of the manifold is related to the deformation tensor β_{ji} as follows. The symmetric part of β_{ji} is the strain tensor

$$e_{ji} = \frac{1}{2}(\beta_{ji} + \beta_{ij}). \tag{2}$$

The rotation of it is the dislocation density tensor

$$\alpha_{ij} = -\frac{1}{2}\epsilon_{ikl}\partial_k\beta_{lj} \tag{3}$$

The incompatibility tensor is, as is well known, derived from the strain tensor

$$J^{ij} = -\epsilon^{ikl}\epsilon^{jmn}\partial_k\partial_m e_{nl} . \tag{4}$$

The dislocation density tensor and the incompatibility tensor are non-divergent

$$\partial_i\alpha_{ij} = 0, \tag{5}$$

$$\partial_i J^{ij} = 0. \tag{6}$$

They are connected by

$$J^{ij} = \epsilon^{ikl}\partial_k\alpha_{jl} + \epsilon^{jkl}\partial_k\alpha_{il} . \tag{7}$$

The strain space is constructed from these quantities. If we define the metric tensor by

$$g_{ji} = \delta_{ji} - 2e_{ji} , \tag{8}$$

and the parameters of connection by

$$\Gamma^{i}_{kj} = -\partial_k\beta_{ji}, \tag{9}$$

we have a distant parallelism strain space. The torsion tensor of the space represents the dislocation distribution

$$S_{klj} = \epsilon_{jkl}\alpha_{ji} \tag{10}$$

and the curvature tensor due to the Levi-Civita parallelism represents the incompatibility

$$K_{ijkl} = \epsilon_{ijm}\epsilon_{kln}J^{nm} \tag{11}$$

The plastic structure of material is completely clarified by the space structure. The correspondence between the physical quantities and the geometrical quantities is shown in Table 1.

3. Dual quantities

Next, let us define the dual physical quantities. By virtue of the plastic defects, some energy is stored in the material. The stored energy E may be expressed in three ways. One is in terms of the residual strain tensor, another is in terms of the dislocation density tensor, and the third is in terms of the incompatibility tensor. Let us assume that an infinitesimal change of the plastic state takes place. Let us denote the corresponding increments of the strain, dislocation density and incompatibility tensors by Δe_{ji}, $\Delta\alpha_{ij}$, and ΔJ^{ij}, respectively. Then the corresponding increment ΔE of the stored energy can be written in the following three ways,

$$\Delta E = \int \sigma^{ij}\Delta e_{ji}\, dX, \tag{12}$$

$$\Delta E = \int \bar{\alpha}^{ij}\Delta\alpha_{ji}\, dX, \tag{13}$$

$$\Delta E = \int \chi_{ji}\Delta J^{ij}\, dX. \tag{14}$$

The quantities σ^{ij}, $\bar{\alpha}_{ij}$ and χ_{ji} represent the physical reactions caused by the existence of e_{ji}, α_{ij} and J^{ij}, respectively. We call these quantities dual quantities. As is well known, σ^{ij} is the stress tensor. We call $\bar{\alpha}_{ij}$ the dual dislocation tensor, and χ_{ji} the dual incompatibility tensor.

The dual quantities are mutually related as follows. On account of (4), we have, integrating (14) twice by parts,

$$\Delta E = \int (-\epsilon^{ikl}\epsilon^{jmn}\partial_k\partial_m\chi_{nl})\Delta e_{ji}\, dX. \tag{15}$$

Hence, we obtain

$$\sigma^{ij} = -\epsilon^{ikl}\epsilon^{jmn}\partial_k\partial_m\chi_{nl}. \tag{16}$$

This shows that the dual incompatibility is nothing but the stress-function tensor. By virtue of the non-divergent character of J^{ij}, χ_{ji} is determined only to within an additive term $\partial_{(i}\varphi_{j)}$, where φ_j is an arbitrary vector.

In a similar way, (13) is transformed to

$$\Delta E = \int(-2\epsilon^{ikl}\partial_k\alpha_{lj})\Delta\alpha_{ji}\, dX, \tag{17}$$

where (7) is taken into account. Therefore, it follows that

$$\bar{\alpha}_{ij} = -2\epsilon^{ikl}\partial_k\alpha_{lj} + \partial_j\psi_i, \tag{18}$$

where ψ_i is another arbitrary vector and the additive term originates from the non-divergent character of α_{ij}.

We call a pair of transformations

$$\left.\begin{array}{l} \chi_{ji} \longrightarrow \chi_{ji} + \partial_{(j}\varphi_{i)}, \\ \bar{\alpha}_{ij} \longrightarrow \bar{\alpha}_{ij} + \partial_j\psi_i, \end{array}\right\}$$

the guage transformation. The guage transformation has no physical meaning so far as the macroscopic stress and strain are concerned. However, some microscopic meaning may be endowed with them, as has been pointed out in the geometrical theory of fatigue fracture [5].

If we restrict the dual quantities to satisfy

$$\partial_j\chi_{ji} = 0, \tag{19}$$

$$\partial_j\bar{\alpha}_{ij} = 0, \tag{20}$$

which we call the Lorentz condition, these quantities are uniquely determined. Moreover

$$\partial_i\bar{\alpha}_{ij} = 0 \tag{21}$$

automatically holds.

Here let us investigate the relation between a primal quantity and the dual. In the case of small disturbances, the stress is linearly related to the strain by

$$\sigma^{ij} = E^{ijkl}e_{kl} \tag{22}$$

where E^{ijkl} is Young's modulus tensor. It is known that, when the

dislocation field α_{ij} is given, the corresponding residual strain field is obtained by the convolution integral

$$e_{ji}(x) = \int a_{jilk}(x-\xi)\alpha_{kl}(\xi)\, d\Xi. \tag{23}$$

The coefficient tensor function $a_{jilk}(x)$ has been obtained by many investigators. Taking (13) into account, we can prove that the dual dislocation is related to the primal dislocation as

$$\bar{\alpha}^{ij}(x) = \int r^{ijkl}(x-\xi)\alpha_{lk}(\xi)\, d\Xi, \tag{24}$$

where r^{ijkl} is a material tensor function defined by

$$r^{tsmn}(x) = \int -E^{ijkl}\, a_{lkmn}(\xi)\, a_{jits}(x+\xi)\, d\Xi. \tag{25}$$

In a similar manner, we can prove that the stress function tensor is related to the incompatibility tensor in the following form

$$\chi_{ji}(x) = \int s_{jikl}(x-\xi)\, J^{lk}(\xi)\, d\Xi. \tag{26}$$

In contrast to the stress-strain relation, these are of non-local character.

The force acting on a dislocation can easily be obtained by using the dual dislocation. Let us consider a dislocation whose direction is denoted by d^j and whose Burgers vector by b^i. Then the force acting on the dislocation is given by

$$F_k = \partial_{[k}\bar{\alpha}_{|i|j]}\, d^j b^i. \tag{27}$$

Hence using the dual quantities, the dynamical problems of plastic continua can more easily be treated.

4. Non-Riemannian stress space

Generalizing the Riemannian stress space, we can construct the non-Riemannian stress space, by which structure the interrelations of the dual quantities are shown. By this stress-space approach, the structures of plastic continua will be investigated further.

Dually to the strain space. let us define the metric tensor g_{ji} by

$$g_{ji} = \delta_{ji} - 2\chi_{ji} \tag{28}$$

and the parameters of connection by

$$\bar{\Gamma}^{\,i}_{kj} = -\partial_k\chi_{ji}. \tag{29}$$

Then, it is easily proved that the torsion tensor $\bar{S}_{kji}$ represents the dual dislocation

$$\bar{S}_{kji} = \frac{1}{4}\epsilon_{kjl}\bar{\alpha}_{li} \qquad (30)$$

and the curvature tensor due to the Levi-Civita parallelism gives the stress tensor

$$\sigma^{ij} = -\epsilon^{ikl}\epsilon^{jmn}\partial_k\partial_m\chi_{nl}. \qquad (31)$$

Thus the interrelations of the dual quantities are geometrized. This space has distant parallelism. Investigating the space structures, we can clarify the dual quantities. The relation between the dual geometrical and physical quantities are given in Table 2. Generalizing the stress space to non-teleparallelism space, to Finslerian space, etc., we shall be able to fortify the theory of palstic continua. We give finally the interrelations of the strain and stress spaces in Table 3.

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Table 1 Strain Space

0	metric g_{ji}	strain e_{ji}
1	torsion S_{kji}	dislocation α_{ij}
2	curvature K_{lkji}	incompatibility J^{ij}
order of differentiation	geometry	physics

Table 2 Stress Space

0	metric g_{ji}	stress function χ_{ji}
1	torsion S_{kji}	dual dislocation $\bar{\alpha}_{ij}$
2	curvature K_{lkji}	stress σ^{ij}
order of differentiation	geometry	physics

Table 3 Interrelation of the Strain and Stress Space